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# Universal Hyperbolic Geometry I: Trigonometry

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**Abstract**

Hyperbolic geometry is developed in a purely algebraic fashion from first principles, without a prior development of differential geometry. The natural connection with the geometry of Lorentz, Einstein and Minkowski comes from a projective point of view, with trigonometric laws that extend to 'points at infinity', here called 'null points', and beyond to 'ideal points' associated to a hyperboloid of one sheet. The theory works over a general field not of characteristic two, and the main laws can be viewed as deformations of those from planar rational trigonometry. There are many new features.

## 1 Introduction

Hyperbolic geometry is set out here in a new and completely algebraic way. This view of the subject, called *universal hyperbolic geometry,* is a special case of the more general geometry described in [20], and has the following characteristics that generally distinguish it from the classical hyperbolic geometry found in for example [1], [3], [4], [7], [8], [10], or from other approaches to the subject, such as [9], [11] or [16].

- a more direct and intimate connection with the geometry of Einstein's Special Theory of Relativity in the framework of Lorentz and Minkowski. In fact *hyperbolic geometry is precisely projective relativistic geometry.* This is a fundamental understanding. The connection with relativistic geometry is also a key feature of [16].
- the basic set-up allows a consistent development of hyperbolic geometry *over the rational numbers.* This is the simplest and purest form of the subject.
- there is a natural development of the subject *over a finite field.* This ties in with work of [2], [12], [14] and others.
- a crucial *duality* between points and lines that connects with, and clarifies, the *pole-polar* duality with respect to the unit circle of projective geometry.
- an unambiguous and concrete treatment of what are traditionally called 'points at infinity', here called *null points,* together with what former generations of projective geometers called 'ideal points' (see for example [13]) which lie on *null lines.* In terms of relativistic geometry, we study the hyperboloid of two sheets with equation $x^2 + y^2 - z^2 = -1$, the null cone with equation $x^2 + y^2 - z^2 = 0$, and the hyperboloid of one sheet with equation $x^2 + y^2 - z^2 = 1$ *together.* The relevance of the latter is also discussed in [15].
- the fundamental metrical measurements of *quadrance* between points and *spread* between lines are similar to the corresponding notions in planar rational trigonometry, and the basic laws of hyperbolic trigonometry may be seen as deformations of those of planar rational trigonometry (see [17]).
- transcendental functions, such as $\log x$, $\sinh x$ or $\cos x$ are not needed. A prior development of the real number system is not needed.
- the existence of a rich *null trigonometry*: trigonometric relations that involve null points and null lines.
- *parallels* play a more specialized role. Somewhat ironically, *Euclid's parallel postulate holds.*

- the isometry group of the geometry *does not act transitively* on the space. The universal hyperbolic plane has aspects which appear in negatively curved Riemannian geometry, other aspects which are Lorentzian, and other aspects which are Euclidean.

- the framework is *algebraic geometry*, rather than *differential geometry*. But it is a form of algebraic geometry that relates more to the historical approach prior to the twentieth century direction. The focus is on *metrical relations* and *concrete polynomial identities* which *encode geometric realities*.

## 1.1 Advantages of the new approach

The advantages of universal hyperbolic geometry over the classical approach include:

- *simplicity and elegance:* the subject is simple enough to be accessible to beginning undergraduates and motivated high school students without a prior understanding of calculus or real numbers. The elementary aspects fit together pleasantly.

- *logical clarity:* traditional treatments of hyperbolic geometry often have obscure foundations, or are rife with arguments that rely on pictorial understanding. The purely algebraic framework frees us from logical difficulties, and allows us to aspire to a complete and unambiguous treatment of the subject from first principles.

- *accuracy:* the new theory achieves much greater accuracy in concrete computations. Many problems can now be solved completely correctly, whereas the classical theory provides only approximate solutions.

- *connections with number theory:* the links between geometrical problems and number–theoretical questions become much more explicit.

- *new directions for special functions:* The remarkable *spread polynomials* of planar rational trigonometry also play a key role in hyperbolic geometry. This family of (almost) orthogonal polynomials replaces the *Chebyshev polynomials of the first kind*.

- *extension of classical geometry:* many more traditional results of Euclidean geometry can now be given their appropriate hyperbolic analogs. Of the thousands of known results in Euclidean geometry, only a fraction currently have analogs in the hyperbolic setting. This turns out to be a consequence of the way we have, up to now, viewed the subject. *Hyperbolic geometry is a richer and ultimately more important theory than Euclidean geometry.*

- *easier constructions:* these are often more direct and elegant than in classical hyperbolic geometry. Over the rational numbers many are even simpler than in Euclidean geometry, in that they require only a base circle and a straightedge.

- *application to inversive geometry:* universal hyperbolic geometry is the natural framework for a comprehensive and general approach to *inversive geometry*.

- *connections with chromogeometry:* universal hyperbolic geometry relates naturally to a new three-fold symmetry in planar geometry that connects Euclidean and relativistic geometries (see [19]).

- *new theorems:* the language and concepts of universal hyperbolic geometry allow us to discover, formulate and prove many new and interesting results.

- *simpler proofs:* the purely algebraic framework allows many proofs to be reduced to algebraic identities which can be easily verified by computer. These identities are often quite interesting in their own right, and warrant further study.

- universal hyperbolic geometry has the same relation to relativistic geometry as spherical or elliptic geometry has to solid Euclidean geometry. Thus this new form of hyperbolic geometry contributes towards a *new geometrical language to study relativistic geometry.*

## 1.2 Note to the reader

This paper represents a major rethinking of this subject, and so careful attention must be given to the basic definitions, some of which are novel, and others which are variants of familiar ones. Some familiarity with rational trigonometry in the Euclidean case is a helpful preliminary. The main reference is [17], see also [21], and the series of YouTube videos called 'WildTrig'.

When we develop geometry seriously, physical intuition and pictorial arguments ought to be separate from the logical structure. To stress the importance of such a logically tight approach, we build up the theory *over a general field*, so there are no pictures. In particular we do not rely on a prior understanding of the continuum, or equivalently the real number system, thus avoiding a major difficulty finessed in most traditional treatments.

Over the rational numbers, the subject nevertheless has a highly visual nature. Section 2.3 describes an extension of the Beltrami Klein model to visualize hyperbolic points and lines over the rational numbers. The reader is also invited to investigate *The Geometer's Sketchpad* worksheets on hyperbolic geometry to be posted at the author's UNSW website:

http://web.maths.unsw.edu.au/˜norman/index.html

Finite prime fields are also highly recommended, as many calculations become simpler, and because this motivates us to think beyond the usual pictures, and connects with interesting combinatorics and graph theory.

The reader will observe that I avoid the use of 'infinite sets'. The reason is simple—I no longer believe they exist. So in the few places where we refer to a field $\mathbb{F}$, we have in mind the specification of a particular *type* of mathematical object, rather than the collection of 'all those of a certain type' into a completed whole. Please be reassured—this has surprisingly little effect on the actual content of the theorems.

Some notational conventions: equality is represented by the symbol $=$ as usual, while the symbol $\equiv$ denotes *specification*, so that for example $A \equiv [3,4]$ is the statement specifying—or *defining*—$A$ to be the ordered pair $[3,4]$. Proofs end in square black boxes, examples end in diamonds. We use the phrase 'precisely when' instead of 'if and only if'. Definitions are in **bold**, while *italics* are reserved for emphasis. Theorems have names, always beginning with a capital letter.

## 1.3 List of theorems

1. Join of points
2. Meet of lines
3. Collinear points
4. Concurrent lines
5. Line through null points
6. Point on null lines
7. Perpendicular point
8. Perpendicular line
9. Opposite points
10. Opposite lines
11. Altitude line
12. Altitude point
13. Parallel line
14. Parallel point
15. Base point
16. Base line
17. Parametrizing a line
18. Parametrizing a point
19. Parametrizing a join
20. Parametrizing a meet
21. Parametrization of null points
22. Parametrization of null lines
23. Join of null points
24. Meet of null lines
25. Null diagonal point
26. Null diagonal line
27. Perpendicular null line
28. Perpendicular null point
29. Parametrizing a null line
30. Parametrizing a null point
31. Triangle trilateral duality
32. Quadrance
33. Zero quadrance
34. Spread
35. Zero spread
36. Quadrance spread duality
37. Quadrance cross ratio
38. Triple quad formula
39. Triple spread formula
40. Complementary quadrances spreads
41. Equal quadrances spreads
42. Pythagoras
43. Pythagoras' dual
44. Spread formula
45. Spread law
46. Spread dual law
47. Quadrea
48. Quadreal
49. Quadrea quadreal product
50. Cross law

51. Cross dual law
52. Triple product relation
53. Triple cross relation
54. Midpoints
55. Midlines
56. Triple quad mid
57. Pythagoras mid
58. Cross mid
59. Couple quadrance spread
60. Three equal quadrances
61. Recursive spreads
62. Thales
63. Thales' dual
64. Right parallax
65. Right parallax dual
66. Napier's rules
67. Napier's dual rules
68. Pons Asinorum
69. Isosceles right
70. Isosceles mid
71. Isosceles triangle
72. Isosceles parallax
73. Equilateral
74. Equilateral mid
75. Triangle proportions
76. Menelaus
77. Menelaus' dual
78. Ceva
79. Ceva's dual
80. Nil cross law
81. Doubly nil triangle
82. Triply nil quadreal
83. Triply nil balance
84. Triply nil orthocenter
85. Triply nil Cevian thinness
86. Triply nil Altitude thinness
87. Singly null singly nil Thales
88. Singly null singly nil orthocenter
89. Null perspective
90. Null subtended quadrance
91. Fully nil quadrangle diagonal
92. 48/64

## 1.4 Overview of contents

Section 2 introduces the main definitions of (*hyperbolic) points* and (*hyperbolic) lines*, *duality* and *perpendicularity*, and establishes elementary but fundamental theorems on *joins* and *meets* and *null points* and *null lines*. The context is in the projective plane associated to a field. Less familiar are the notions of side and vertex, which are here given rather different definitions—a *side* is basically an unordered pair of points, while a *vertex* is an unordered pair of lines. The combination of a point and a line is called a *couple*. Various important constructions arise from studying these combinations which are prior to the investigations of *triangles* (a set of three non-collinear points) and *trilaterals* (a set of three non-concurrent lines). Null points and null lines are parametrized, and in terms of these parametrizations descriptions of joins and meets of null points and null lines are given.

Section 3 introduces the main metrical notions of *quadrance between points* and *spread between lines*. These are dual notions. Both can be described purely in terms of projective geometry and cross ratios. The main laws of hyperbolic trigonometry are established: the *Triple quad formula* and its dual the *Triple spread formula*, *Pythagoras' theorem* and its dual, the *Spread law*, which is essentially self dual, and the *Cross law* and its dual. The important notion of the *quadrea* of a triangle and dually the *quadreal* of a trilateral are closely related to the Cross law, which as in the planar case is the most important of the trigonometric laws. Various theorems have simpler formulations in the case when sides have midpoints or vertices have midlines (which play the role of angle bisectors). Finally the spread polynomials are briefly introduced.

Section 4 studies particular special types of triangles and trilaterals, first right triangles with the parallax theorem and Napier's rules and its dual. Isosceles and equilateral triangles are studied. Various triangle proportions theorems are established, such as Menelaus' theorem and Ceva's theorem and their duals.

Section 5 introduces the rich subject of null trigonometry, concerned with special relationships involving null points and null lines. These results are often new, and sometimes spectacular. In particular we establish two aspects of the thinness of triangles, some results on subtended quadrances and spreads, and finally the curious 48/64 theorem on the diagonal spreads of a completely nil quadrangle.

## 1.5 Thanks

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# 2 Hyperbolic points and lines

This section discusses *proportions*, and then introduces the main objects of *(hyperbolic) points* and *(hyperbolic) lines* and the relations of *duality* and *perpendicularity*. We briefly describe a visual model, then establish fundamental facts about *joins, meets, collinearity* and *concurrency*. Points and lines are parametrized in two different ways, and then we introduce the important notions of *side, vertex* and *couple*. These definitions will likely be novel to the reader. Some canonical constructions that arise from them are investigated. *Null points* and *null lines* are parametrized and formulas for their meets and joins obtained. Finally the dual notions of *triangle* and *trilateral* are introduced.

## 2.1 Proportions

We work over a field $\mathbb{F}$, not of characteristic two, whose elements are called **numbers**. Those readers who are not comfortable with the general notion of field may restrict themselves to the field of rational numbers $\mathbb{Q}$. In fact one of the remarkable consequences of this development is that there is a *complete theory of hyperbolic geometry over the rational numbers.*

A 2-**proportion** $x : y$ is an ordered pair of numbers $x$ and $y$, *not both zero*, with the convention that for any non-zero number $\lambda$

$$x : y = \lambda x : \lambda y.$$

This may be restated by saying that

$$x_1 : y_1 = x_2 : y_2$$

precisely when

$$x_1 y_2 - x_2 y_1 = 0.$$

A 3-**proportion** $x : y : z$ is an ordered triple of numbers $x, y$ and $z$, *not all zero*, with the convention that for any non-zero number $\lambda$

$$x : y : z = \lambda x : \lambda y : \lambda z.$$

This may be restated by saying that

$$x_1 : y_1 : z_1 = x_2 : y_2 : z_2$$

precisely when the following three conditions hold:

$$x_1 y_2 - x_2 y_1 = 0 \qquad y_1 z_2 - y_2 z_1 = 0 \qquad z_1 x_2 - z_2 x_1 = 0. \tag{1}$$

If the context is clear, we just refer to *proportions*, instead of 2-proportions or 3-proportions.

## 2.2 Points, lines, duality and perpendicularity

A **(hyperbolic) point** is a 3-proportion $a \equiv [x : y : z]$ enclosed in square brackets. A **(hyperbolic) line** is a 3-proportion $L \equiv (l : m : n)$ enclosed in round brackets.

The point $a \equiv [x : y : z]$ is **dual** to the line $L \equiv (l : m : n)$ precisely when

$$x : y : z = l : m : n.$$

In this case we write $a^\perp = L$ or $L^\perp = a$. Then

$$\left(a^\perp\right)^\perp = a \qquad \text{and} \qquad \left(L^\perp\right)^\perp = L.$$

Each point is dual to exactly one line, and conversely. We will often, but not always, use the notational convention of corresponding *small letters* for *points* and *capital letters* for *dual lines*, for example $a$ for a point and $A \equiv a^\perp$ for the dual line, or $L$ for a line and $l \equiv L^\perp$ for the dual point.

The point $a \equiv [x : y : z]$ **lies on** the line $L \equiv (l : m : n)$, or equivalently $L$ **passes through** $a$, precisely when

$$lx + my - nz = 0.$$

The point $a \equiv [x : y : z]$ is **null** precisely when it lies on its dual line, in other words when

$$x^2 + y^2 - z^2 = 0.$$

The line $L \equiv (l : m : n)$ is **null** precisely when it passes through its dual point, in other words when

$$l^2 + m^2 - n^2 = 0.$$

The dual of a null point is a null line and conversely.

Points $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ are **perpendicular** precisely when

$$x_1 x_2 + y_1 y_2 - z_1 z_2 = 0.$$

This is equivalent to the condition that $a_1$ lies on $a_2^{\perp}$, or that $a_2$ lies on $a_1^{\perp}$.

Lines $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$ are **perpendicular** precisely when

$$l_1 l_2 + m_1 m_2 - n_1 n_2 = 0.$$

This is equivalent to the condition that $L_1$ passes through $L_2^{\perp}$, or that $L_2$ passes through $L_1^{\perp}$.

*There is a complete duality in the theory between points and lines.* Any result thus has a corresponding dual result, obtained by interchanging the roles of points and lines. We call this the *duality principle*, and use it often to eliminate repetition of statements and proofs of theorems. To maintain this principle, *we treat points and lines symmetrically*, especially initially. Later in the paper we leave the formulation of dual statements to the reader.

We denote by $\mathbb{F}^3$ the three-dimensional space of **vectors** $v \equiv (x, y, z)$. If $v \equiv (x, y, z)$ has coordinates which are not all zero, then $[v] \equiv [x : y : z]$ denotes the corresponding (hyperbolic) point, and $(v) \equiv (x : y : z)$ denotes the corresponding (hyperbolic) line.

## 2.3 Visualizing universal hyperbolic geometry

Although not logically necessary, let's point out one way of visualizing the subject. This is essentially the Beltrami Klein view, but extended beyond the unit disk, and with underlying field the rational numbers $\mathbb{Q}$. Think of a point $a \equiv [x : y : z]$ as representing the central line (one-dimensional subspace) in three-dimensional space $\mathbb{Q}^3$ through the origin and the point $[x, y, z]$. Think of the line $L \equiv (l : m : n)$ as being the central plane (two-dimensional subspace) with equation $lx + my - nz = 0$. So the notion '$a$ lies on $L$' has the usual interpretation.

Both points and lines are projective objects, and may be illustrated in diagrams using their meets with the plane $z = 1$ in the usual way. So the (hyperbolic) point $a \equiv [x : y : z]$ becomes generically the planar point

$$[X, Y] \equiv \left[\frac{x}{z}, \frac{y}{z}\right]$$

while the (hyperbolic) line $L \equiv (l : m : n)$ becomes generically the planar line with equation

$$lX + mY = n.$$

Otherwise in case $z = 0$, or $l = m = 0$, we have respectively points at infinity and a line at infinity, represented in the usual way by directions in the $z = 1$ plane.

Null points become planar points on the unit circle $X^2 + Y^2 = 1$. (Over other fields, there may also be two null points at infinity, but this does not happen over the rational numbers). Null lines become tangent lines to this unit circle. Both the interior and exterior of the unit circle are important, the former corresponding to hyperboloids of two sheets, the latter to hyperboloids of one sheet in the ambient three-dimensional relativistic space. Neither has preference over the other. In fact *the most important objects are the null points and null lines.*

The duality between points and lines becomes the pole-polar duality of projective geometry, since the planar point $[a, b]$ is the pole of the planar line $aX + bY = 1$ with respect to the unit circle $X^2 + Y^2 = 1$. Perpendicularity then becomes a straightedge construction, since the pole of a line or the polar of a point may be so constructed, once the unit circle is given. The metrical structure comes from the quadratic form $x^2 + y^2 - z^2$ in the ambient space.

For an interesting variant, represent the projective plane by intersecting the three-dimensional space $\mathbb{Q}^3$ with the plane $x = 1$. This provides a more 'hyperbolic view'; less familiar, but also worthy of study.

Let's emphasize that the following development of the subject is logically independent of any one visual interpretation of it.

## 2.4 Joins and meets

**Theorem 1 (Join of points)** *If $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ are distinct points, then there is exactly one line $L$ which passes through them both, namely*

$$L \equiv a_1 a_2 \equiv (y_1 z_2 - y_2 z_1 : z_1 x_2 - z_2 x_1 : x_2 y_1 - x_1 y_2).$$

**Proof.** The 3-proportion

$$y_1 z_2 - y_2 z_1 : z_1 x_2 - z_2 x_1 : x_2 y_1 - x_1 y_2 \tag{2}$$

is well-defined, since if we multiply the coordinates of either $a_1$ or $a_2$ by a non-zero number, then each term in (2) is correspondingly changed, and since $a_1$ and $a_2$ are distinct, at least one of the three terms is non-zero from (1).

The line

$$L \equiv (y_1 z_2 - y_2 z_1 : z_1 x_2 - z_2 x_1 : x_2 y_1 - x_1 y_2)$$

passes through both $a_1$ and $a_2$, since

$$(y_1 z_2 - y_2 z_1)\, x_1 + (z_1 x_2 - z_2 x_1)\, y_1 - (x_2 y_1 - x_1 y_2)\, z_1 = 0$$
$$(y_1 z_2 - y_2 z_1)\, x_2 + (z_1 x_2 - z_2 x_1)\, y_2 - (x_2 y_1 - x_1 y_2)\, z_2 = 0.$$

The 3-proportions $x_1 : y_1 : z_1$ and $x_2 : y_2 : z_2$ are by assumption unequal, so the system of equations

$$lx_1 + my_1 - nz_1 = 0$$
$$lx_2 + my_2 - nz_2 = 0$$

has up to a multiple exactly one solution, showing that $L$ is unique. ■

The line $L \equiv a_1 a_2$ is the **join** of the points $a_1$ and $a_2$.

**Theorem 2 (Meet of lines)** *If $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$ are distinct lines, then there is exactly one point $a$ which lies on them both, namely*

$$a \equiv L_1 L_2 \equiv [m_1 n_2 - m_2 n_1 : n_1 l_2 - n_2 l_1 : l_2 m_1 - l_1 m_2].$$

**Proof.** This is dual to the Join of points theorem. ■

The point $a \equiv L_1 L_2$ is the **meet** of the lines $L_1$ and $L_2$. Note that for any distinct points $a_1$ and $a_2$

$$(a_1 a_2)^{\perp} = a_1^{\perp} a_2^{\perp}.$$

Similarly for any distinct lines $L_1$ and $L_2$

$$(L_1 L_2)^{\perp} = L_1^{\perp} L_2^{\perp}.$$

Both of the previous theorems involve implicitly the **hyperbolic cross product function**

$$J(x_1, y_1, z_1; x_2, y_2, z_2) \equiv (y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_2 y_1 - x_1 y_2).$$

This is a hyperbolic version of the more familiar *Euclidean cross product*, and it enjoys many of the same properties. It is ubiquitous in the two-dimensional hyperbolic geometry developed in this paper, and many computations amount essentially to repeated evaluations of this function.

## 2.5 Collinear points and concurrent lines

Three or more points which lie on a common line are **collinear**. Three or more lines which pass through a common point are **concurrent**.

**Theorem 3 (Collinear points)** *The points $a_1 \equiv [x_1 : y_1 : z_1]$, $a_2 \equiv [x_2 : y_2 : z_2]$ and $a_3 \equiv [x_3 : y_3 : z_3]$ are collinear precisely when*

$$x_1 y_2 z_3 - x_1 y_3 z_2 + x_2 y_3 z_1 - x_3 y_2 z_1 + x_3 y_1 z_2 - x_2 y_1 z_3 = 0.$$

**Proof.** If two of the three points are distinct, say $a_1$ and $a_2$, then by the Join of points theorem

$$a_1a_2 = (y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_2y_1 - x_1y_2)$$

and $a_3$ lies on $a_1a_2$ precisely when

$$(y_1z_2 - y_2z_1)\, x_3 + (z_1x_2 - z_2x_1)\, y_3 - (x_2y_1 - x_1y_2)\, z_3 = 0.$$

This is the condition of the theorem. If all points are identical, then they are collinear, and the expression

$$x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_3y_2z_1 + x_3y_1z_2 - x_2y_1z_3$$

is by symmetry zero. ■

**Theorem 4 (Concurrent lines)** *The lines* $L_1 \equiv (l_1 : m_1 : n_1)$, $L_2 \equiv (l_2 : m_2 : n_2)$ *and* $L_3 \equiv (l_3 : m_3 : n_3)$ *are concurrent precisely when*

$$l_1m_2n_3 - l_1m_3n_2 + l_2m_3n_1 - l_3m_2n_1 + l_3m_1n_2 - l_2m_1n_3 = 0.$$

**Proof.** This is dual to the Collinear points theorem. ■

The formulas of the two previous theorems can be recast as the determinantal equations

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0 \qquad \text{and} \qquad \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0.$$

**Theorem 5 (Line through null points)** *Any line* $L$ *passes through at most two null points.*

**Proof.** If $L \equiv (l : m : n)$ then any null point $\alpha \equiv [x : y : z]$ lying on $L$ satisfies

$$lx + my - nz = 0 \tag{3}$$

and

$$x^2 + y^2 - z^2 = 0. \tag{4}$$

After substituting (3) into (4) you get a homogeneous quadratic equation in two variables. This has at most two solutions up to a factor, so there are at most two solutions $x : y : z$ to this pair of equations. ■

**Theorem 6 (Point on null lines)** *Any point* $a$ *lies on at most two null lines.*

**Proof.** This is dual to the Line through null points theorem. ■

The two previous theorems lead naturally to an important characterization of non-null points and lines, in terms of their relations with null points and lines. A non-null point $a$ is defined to be **internal** precisely when it lies on no null lines, and is **external** precisely when it lies on two null lines. A non-null line $L$ is defined to be **internal** precisely when it passes through two null points, and is **external** precisely when it passes through no null points. This way points and lines are either *internal*, *null* or *external*.

These notions play a major role when we discuss *isometries* in a future paper, but they are not necessary for the development of basic trigonometry, indeed the main theorems of trigonometry apply equally to both internal and external points and lines. So we will not develop these concepts further in this paper.

## 2.6 Sides and vertices

A **side** $\overline{a_1a_2}$ is a set $\{a_1, a_2\}$ of two points. A **vertex** $\overline{L_1L_2}$ is a set $\{L_1, L_2\}$ of two lines. Clearly

$$\overline{a_1a_2} = \overline{a_2a_1} \qquad \text{and} \qquad \overline{L_1L_2} = \overline{L_2L_1}.$$

If $\overline{a_1a_2}$ is a side, then $a_1a_2$ is the **line** of the side. The side $\overline{a_1a_2}$ is a **null side** precisely when $a_1a_2$ is a null line. The side $\overline{a_1a_2}$ is a **nil side** precisely when at least one of $a_1$ or $a_2$ is a null point. In this case it is a **singly-nil side** precisely when exactly one of $a_1$ or $a_2$ is a null point, and a **doubly-nil side** precisely when both $a_1$ and $a_2$ are null points.

If $\overline{L_1L_2}$ is a vertex, then $L_1L_2$ is the **point** of the vertex. The vertex $\overline{L_1L_2}$ is a **null vertex** precisely when $L_1L_2$ is a null point. The vertex $\overline{L_1L_2}$ is a **nil vertex** precisely when at least one of $L_1$ or $L_2$ is a null line. In this case it is a **singly-nil vertex** precisely when exactly one of $L_1$ or $L_2$ is a null line, and a **doubly-nil vertex** precisely when both $L_1$ and $L_2$ are null lines.

The **dual** of the side $\overline{a_1a_2}$ is the vertex $\overline{a_1^\perp a_2^\perp}$. The **dual** of the vertex $\overline{L_1L_2}$ is the side $\overline{L_1^\perp L_2^\perp}$.

The side $\overline{a_1a_2}$ is a **right side** precisely when $a_1$ is perpendicular to $a_2$. The vertex $\overline{L_1L_2}$ is a **right vertex** precisely when $L_1$ is perpendicular to $L_2$.

**Theorem 7 (Perpendicular point)** *For any side $\overline{a_1a_2}$ there is a unique point $p$ which is perpendicular to both $a_1$ and $a_2$, namely*

$$p \equiv a_1^\perp a_2^\perp = (a_1a_2)^\perp .$$

**Proof.** Any point $p$ which is perpendicular to both $a_1$ and $a_2$ must lie on both $a_1^\perp$ and $a_2^\perp$, and since $a_1$ and $a_2$ are distinct, the Meet of lines theorem asserts that there is exactly one such point, namely $p \equiv a_1^\perp a_2^\perp = (a_1a_2)^\perp$. ∎

The point $p$ is the **perpendicular point** of $\overline{a_1a_2}$. It may happen that $p$ lies on $a_1a_2$; this will occur precisely when $\overline{a_1a_2}$ is a null side.

**Theorem 8 (Perpendicular line)** *For any vertex $\overline{L_1L_2}$ there is a unique line $P$ which is perpendicular to both $L_1$ and $L_2$, namely*

$$P \equiv L_1^\perp L_2^\perp = (L_1L_2)^\perp .$$

**Proof.** This is dual to the Perpendicular point theorem. ∎

The line $P$ is the **perpendicular line** of $\overline{L_1L_2}$. It may happen that $P$ passes through $L_1L_2$; this will occur precisely when $\overline{L_1L_2}$ is a null vertex.

**Example 1** *Consider the distinct points $a_1 \equiv [x:0:1]$ and $a_2 \equiv [0:y:1]$. Then $a_1a_2 = (y:x:xy)$ and so the perpendicular point of the side $\overline{a_1a_2}$ is $p = [y:x:xy]$.* ⋄

**Example 2** *Consider the distinct lines $L_1 \equiv (l_1:m_1:0)$ and $L_2 \equiv (l_2:m_2:0)$. Then $L_1L_2 = [0:0:1]$ and so the perpendicular line of the vertex $\overline{L_1L_2}$ is $P = (0:0:1)$.* ⋄

**Theorem 9 (Opposite points)** *For any non-null side $\overline{a_1a_2}$ there is a unique point $o_1$ which lies on $a_1a_2$ and is perpendicular to $a_1$, namely*

$$o_1 \equiv (a_1a_2)\, a_1^\perp,$$

*and there is a unique point $o_2$ which lies on $a_1a_2$ and is perpendicular to $a_2$, namely*

$$o_2 \equiv (a_1a_2)\, a_2^\perp.$$

*The points $o_1$ and $o_2$ are distinct. If $a_1$ is a null point, then $o_1 = a_1$, and if $a_2$ is a null point, then $o_2 = a_2$. If $a_1 \equiv [v_1]$ and $a_2 \equiv [v_2]$ with vectors $v_1 \equiv (x_1, y_1, z_1)$ and $v_2 \equiv (x_2, y_2, z_2)$, then*

$$\begin{aligned} o_1 &= \left[(x_1x_2 + y_1y_2 - z_1z_2)\, v_1 - \left(x_1^2 + y_1^2 - z_1^2\right) v_2\right] \\ o_2 &= \left[\left(x_2^2 + y_2^2 - z_2^2\right) v_1 - (x_1x_2 + y_1y_2 - z_1z_2)\, v_2\right] . \end{aligned}$$

**Proof.** If $a_1a_2 = a_1^\perp$ then $a_1$ lies on its dual $a_1^\perp$, so that $a_1$ is a null point, and so $a_1^\perp = a_1a_2$ is a null line. Since we assume that $\overline{a_1a_2}$ is a non-null side, we conclude that $a_1^\perp$ is distinct from $a_1a_2$, so that $o_1 \equiv (a_1a_2)\, a_1^\perp$ is well-defined, and is the unique point lying on $a_1a_2$ which is perpendicular to $a_1$. Similarly $o_2 \equiv (a_1a_2)\, a_2^\perp$ is the unique point lying on $a_1a_2$ which is perpendicular to $a_2$.

If $o_1 = o_2$ then $o_1 = a_1^\perp a_2^\perp = (a_1a_2)^\perp$ lies on its dual $a_1a_2$, which would imply that $a_1a_2$ is a null line, which contradicts the assumption that $\overline{a_1a_2}$ is a non-null side. So $o_1$ and $o_2$ are distinct.

If $a_1$ is a null point, then $a_1$ lies on both $a_1^\perp$ and $a_1a_2$, so $o_1 = a_1$. Similarly if $a_2$ is a null point then $o_2 = a_2$.

Now suppose that $a_1 \equiv [v_1]$ and $a_2 \equiv [v_2]$ where $v_1 \equiv (x_1, y_1, z_1)$ and $v_2 \equiv (x_2, y_2, z_2)$ are vectors. Then from the Join of points theorem and the Meet of lines theorem,

$$a_1a_2 = (y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_2y_1 - x_1y_2)$$

and

$$\begin{aligned}
&(a_1 a_2)\, a_1^{\perp} \\
&= \left[ \begin{array}{c} (z_1 x_2 - z_2 x_1)\, z_1 - y_1\, (x_2 y_1 - x_1 y_2) : (x_2 y_1 - x_1 y_2)\, x_1 - z_1\, (y_1 z_2 - y_2 z_1) \\ : x_1\, (z_1 x_2 - z_2 x_1) - (y_1 z_2 - y_2 z_1)\, y_1 \end{array} \right] \\
&= \left[ x_1 y_1 y_2 - x_1 z_1 z_2 - x_2 y_1^2 + x_2 z_1^2 : x_1 x_2 y_1 - y_1 z_1 z_2 - x_1^2 y_2 + y_2 z_1^2 : x_1 x_2 z_1 + y_1 y_2 z_1 - x_1^2 z_2 - y_1^2 z_2 \right].
\end{aligned}$$

On the other hand

$$\begin{aligned}
&(x_1 x_2 + y_1 y_2 - z_1 z_2)\,(x_1, y_1, z_1) - \left(x_1^2 + y_1^2 - z_1^2\right)(x_2, y_2, z_2) \\
&= \left(x_1 y_1 y_2 - x_1 z_1 z_2 - x_2 y_1^2 + x_2 z_1^2, x_1 x_2 y_1 - y_1 z_1 z_2 - x_1^2 y_2 + y_2 z_1^2, x_1 x_2 z_1 + y_1 y_2 z_1 - x_1^2 z_2 - y_1^2 z_2\right).
\end{aligned}$$

So

$$o_1 \equiv (a_1 a_2)\, a_1^{\perp} = \left[ (x_1 x_2 + y_1 y_2 - z_1 z_2)\, v_1 - \left(x_1^2 + y_1^2 - z_1^2\right) v_2 \right].$$

Interchanging the indices gives

$$o_2 \equiv (a_1 a_2)\, a_2^{\perp} = \left[ \left(x_2^2 + y_2^2 - z_2^2\right) v_1 - (x_1 x_2 + y_1 y_2 - z_1 z_2)\, v_2 \right]. \quad \blacksquare$$

The points $o_1$ and $o_2$ are the **opposite points** of the side $\overline{a_1 a_2}$. Since $o_1$ and $o_2$ are distinct, the side $\overline{o_1 o_2}$ is well-defined, and non-null since $o_1 o_2 = a_1 a_2$. The side $\overline{o_1 o_2}$ is **opposite** to the side $\overline{a_1 a_2}$. This relationship is involutory: $\overline{a_1 a_2}$ is also opposite to $\overline{o_1 o_2}$.

**Theorem 10 (Opposite lines)** *For any non-null vertex $\overline{L_1 L_2}$ there is a unique line $O_1$ which passes through $L_1 L_2$ and is perpendicular to $L_1$, namely*

$$O_1 \equiv (L_1 L_2)\, L_1^{\perp},$$

*and there is a unique line $O_2$ which passes through $L_1 L_2$ and is perpendicular to $L_2$, namely*

$$O_2 \equiv (L_1 L_2)\, L_2^{\perp}.$$

*The lines $O_1$ and $O_2$ are distinct. If $L_1$ is a null line, then $O_1 = L_1$, and if $L_2$ is a null line, then $O_2 = L_2$. If $L_1 = (v_1)$ and $L_2 = (v_2)$ with vectors $v_1 = (l_1, m_1, n_1)$ and $v_2 = (l_2, m_2, n_2)$, then*

$$\begin{aligned}
O_1 &= \left( (l_1 l_2 + m_1 m_2 - n_1 n_2)\, v_1 - \left(l_1^2 + m_1^2 - n_1^2\right) v_2 \right) \\
O_2 &= \left( \left(l_2^2 + m_2^2 - n_2^2\right) v_1 - (l_1 l_2 + m_1 m_2 - n_1 n_2)\, v_2 \right).
\end{aligned}$$

**Proof.** This is dual to the Opposite points theorem. $\blacksquare$

The lines $O_1$ and $O_2$ are the **opposite lines** of the vertex $\overline{L_1 L_2}$. Since $O_1$ and $O_2$ are distinct, the vertex $\overline{O_1 O_2}$ is well-defined, and non-null since $O_1 O_2 = A_1 A_2$. The vertex $\overline{O_1 O_2}$ is **opposite** to the vertex $\overline{A_1 A_2}$. This relationship is involutory: $\overline{L_1 L_2}$ is also opposite to $\overline{O_1 O_2}$.

If $L_1 = a_1^{\perp}$ and $L_2 = a_2^{\perp}$, then the opposite points $o_1$ and $o_2$ of $\overline{a_1 a_2}$ are dual respectively to the opposite lines $O_1$ and $O_2$ of $\overline{L_1 L_2}$.

**Example 3** *Consider the distinct points $a_1 \equiv [x : 0 : 1]$ and $a_2 \equiv [0 : y : 1]$. Then the opposite points of $\overline{a_1 a_2}$ are*

$$o_1 = \left[ x : y\left(x^2 - 1\right) : x^2 \right] \qquad \textit{and} \qquad o_2 = \left[ x\left(y^2 - 1\right) : y : y^2 \right]. \quad \diamond$$

**Example 4** *Consider the distinct lines $L_1 \equiv (l_1 : m_1 : 0)$ and $L_2 \equiv (l_2 : m_2 : 0)$. Then the opposite lines of $\overline{L_1 L_2}$ are*

$$O_1 = (-m_1 : l_1 : 0) \qquad \textit{and} \qquad O_2 = (-m_2 : l_2 : 0). \quad \diamond$$

## 2.7 Couples

A **couple** $\overline{aL}$ is a set $\{a, L\}$ consisting of a point $a$ and a line $L$ such that $a$ does not lie on $L$. The **dual** of the couple $\overline{aL}$ is the couple $\overline{L^{\perp} a^{\perp}}$. The couple $\overline{aL}$ is dual to itself precisely when $a$ is dual to $L$, in which case we say $\overline{aL}$ is a **dual couple**. A couple $\overline{aL}$ is **null** precisely when $a$ is a null point or $L$ is a null line (or both).

**Theorem 11 (Altitude line)** *For any non-dual couple $\overline{aL}$ there is a unique line $N$ which passes through $a$ and is perpendicular to $L$, namely*

$$N \equiv aL^{\perp}.$$

**Proof.** Any line $N$ which passes through $a$ and is perpendicular to $L$ must also pass through $L^{\perp}$, and since $a$ and $L^{\perp}$ are by assumption distinct there is exactly one such line, namely $N \equiv aL^{\perp}$. ■

The line $N \equiv aL^{\perp}$ is the **altitude line** of $\overline{aL}$, or the **altitude line to $L$ through** $a$.

**Theorem 12 (Altitude point)** *For any non-dual couple $\overline{aL}$ there is a unique point $n$ which lies on $L$ and is perpendicular to $a$, namely*

$$n \equiv a^{\perp}L.$$

**Proof.** This is dual to the Altitude line theorem. ■

The point $n \equiv a^{\perp}L$ is the **altitude point of** $\overline{aL}$, or the **altitude point to $a$ on** $L$. The altitude line $N$ and the altitude point $n$ of $\overline{aL}$ are dual.

If $a \equiv [x : y : z]$ and $L \equiv (k : l : m)$, then the coefficients of $N$ and $n$ are both given by $J(x, y, z; k, l, m)$.

**Theorem 13 (Parallel line)** *For any non-dual couple $\overline{aL}$ there is a unique line $R$ which passes through $a$ and is perpendicular to the altitude line $N$ of $\overline{aL}$, namely*

$$R \equiv a\left(a^{\perp}L\right).$$

**Proof.** The line $R \equiv a\left(a^{\perp}L\right)$ is well-defined since $a$ does not lie on $L$. Furthermore $R$ passes through $a$ and is perpendicular to $N \equiv aL^{\perp}$, and any such line must be $a\left(a^{\perp}L\right)$. ■

The line $R \equiv a\left(a^{\perp}L\right)$ is the **parallel line** of $\overline{aL}$, or the **parallel line to $L$ through** $a$. This is the only notion of parallel line that appears in this treatment of the subject. Note that parallel lines are only defined *relative to a couple*, that is a point and a line, and *are unique*. Ironically, Euclid's parallel postulate is alive and well in universal hyperbolic geometry!

**Theorem 14 (Parallel point)** *For any non-dual couple $\overline{aL}$ there is a unique point $r$ which lies on $a^{\perp}$ and is perpendicular to the altitude point $n$ of $\overline{aL}$, namely*

$$r \equiv a^{\perp}\left(aL^{\perp}\right).$$

**Proof.** This is dual to the Parallel line theorem. ■

The point $r \equiv a^{\perp}\left(aL^{\perp}\right)$ is the **parallel point** of $\overline{aL}$, or the **parallel point to $a$ on** $L$. The parallel line $R$ and the parallel point $r$ of $\overline{aL}$ are dual.

**Theorem 15 (Base point)** *For any non-dual couple $\overline{aL}$ there is a unique point $b$ which lies on both $L$ and the altitude line $N$ of $\overline{aL}$, namely*

$$b \equiv \left(aL^{\perp}\right)L.$$

**Proof.** The point $b \equiv \left(aL^{\perp}\right)L$ is well-defined since $L$ does not pass through $a$. Furthermore $b$ lies on both $L$ and $N \equiv aL^{\perp}$, and any such point must be $\left(aL^{\perp}\right)L$. ■

The point $b \equiv \left(aL^{\perp}\right)L$ is the **base point** of $\overline{aL}$, or the **base point of the altitude line to $L$ through** $a$.

**Theorem 16 (Base line)** *For any non-dual couple $\overline{aL}$ there is a unique line $B$ which passes through both $L^{\perp}$ and the altitude point $n$ of $\overline{aL}$, namely*

$$B \equiv \left(a^{\perp}L\right)L^{\perp}.$$

**Proof.** This is dual to the Base point theorem. ■

The line $B \equiv \left(a^{\perp}L\right)L^{\perp}$ is the **base line** of $\overline{aL}$, or the **base line of the altitude point to $a$ on** $L$. The base line $B$ and the base point $b$ of $\overline{aL}$ are dual.

Can one continue in this fashion to create more and more points and lines associated to a couple $\overline{aL}$? If one restricts to using only the notions introduced so far—essentially perpendicularity—the answer is generally *no—the altitudes, parallels and bases* of a couple exhaust such constructions. It is a good exercise to confirm this.

## 2.8 Parametrizing points and lines

**Theorem 17 (Parametrizing a line)** *If a point $a$ lies on a line $L \equiv (l : m : n)$, then there are numbers $p, r$ and $s$ such that*

$$a = [np - ms : ls + nr : lp + mr] .$$

**Proof.** Suppose first that at least two of $l, m$ and $n$ are non-zero. The points $[n : 0 : l]$, $[0 : n : m]$ and $[-m : l : 0]$ are then distinct, and all lie on $L$. So the equation

$$lx + my - nz = 0 \tag{5}$$

has non-zero solutions $(n, 0, l)$, $(0, n, m)$ and $(-m, l, 0)$. Since

$$\begin{pmatrix} n & 0 & l \\ 0 & n & m \\ -m & l & 0 \end{pmatrix}$$

is a rank two matrix, every non-zero solution to (5) has the form

$$\begin{aligned} (x, y, z) &= p\,(n, 0, l) + r\,(0, n, m) + s\,(-m, l, 0) \\ &= (np - ms, ls + nr, lp + mr) \end{aligned}$$

for some numbers $p, r$ and $s$. So if $a \equiv [x : y : z]$ lies on $L$, it can be written as $a = [np - ms : ls + nr : lp + mr]$.

If only one of $l, m$ and $n$ are non-zero, say $n \neq 0$, then every point $a$ lying on $L$ has the form $[x : y : 0]$, which can be written as $[np : nr : 0]$, and this is also of the required form. ■

**Theorem 18 (Parametrizing a point)** *If a line $L$ passes through a point $a \equiv [x : y : z]$, then there are numbers $p, r$ and $s$ such that*

$$L = (zp - ys : xs + zr : xp + yr) .$$

**Proof.** This is dual to the Parametrizing a line theorem. ■

If we know that a line is a join of two particular points, then it is often more convenient to parametrize it in a different fashion. We frame this is in the language of vectors.

**Theorem 19 (Parametrizing a join)** *Suppose that $v_1$ and $v_2$ are linearly independent vectors, with $a_1 \equiv [v_1]$ and $a_2 \equiv [v_2]$ the corresponding points. Then for any point $a$ lying on $a_1 a_2$, there is a unique proportion $t : u$ such that*

$$a = [tv_1 + uv_2] .$$

**Proof.** Suppose that $v_1 \equiv (x_1, y_1, z_1)$ and $v_2 \equiv (x_2, y_2, z_2)$. Since $v_1$ and $v_2$ are linearly independent, $a_1 \equiv [v_1]$ and $a_2 \equiv [v_2]$ are distinct points. By the Join of points theorem,

$$a_1 a_2 = (y_1 z_2 - y_2 z_1 : z_1 x_2 - z_2 x_1 : x_2 y_1 - x_1 y_2)$$

so any point $a \equiv [x : y : z]$ lying on $a_1 a_2$ satisfies

$$(y_1 z_2 - y_2 z_1)\, x + (z_1 x_2 - z_2 x_1)\, y - (x_2 y_1 - x_1 y_2)\, z = 0.$$

As an equation for $v \equiv (x, y, z)$, this has independent solutions $v_1$ and $v_2$, so any non-zero solution is a non-zero linear combination $v = tv_1 + uv_2$. Then

$$a = [v] = [tv_1 + uv_2]$$

where the proportion $t : u$ is unique. ■

**Theorem 20 (Parametrizing a meet)** *Suppose that $v_1$ and $v_2$ are linearly independent vectors, with $L_1 \equiv (v_1)$ and $L_2 \equiv (v_2)$ the corresponding lines. Then for any line $L$ passing through $L_1 L_2$, there is a unique proportion $t : u$ such that*

$$L = (tv_1 + uv_2) .$$

**Proof.** This is dual to the Parametrizing a join theorem. ■

### 2.9 Null points and null lines

Null points and null lines are of central importance in universal hyperbolic geometry. We first parametrize them, using Pythagorean triples.

**Theorem 21 (Parametrization of null points)** *Any null point $\alpha$ is of the form*

$$\alpha = \alpha\,(t:u) \equiv \left[t^2 - u^2 : 2tu : t^2 + u^2\right]$$

*for some unique proportion $t:u$.*

**Proof.** The identity

$$\left(t^2 - u^2\right)^2 + (2tu)^2 - \left(t^2 + u^2\right)^2 = 0$$

ensures that every point of the form $\alpha\,(t:u)$ is null.

Suppose that $\alpha \equiv [x:y:z]$ is a null point, so that $x^2 + y^2 - z^2 = 0$. If $x + z \neq 0$, then set

$$t \equiv x + z \qquad \text{and} \qquad u \equiv y$$

so that

$$\begin{aligned}\left[t^2 - u^2 : 2tu : t^2 + u^2\right] &= \left[x^2 + 2xz + z^2 - y^2 : 2xy + 2yz : x^2 + 2xz + z^2 + y^2\right] \\ &= \left[2xz + 2x^2 : 2xy + 2yz : 2xz + 2z^2\right] \\ &= \left[2x\,(x+z) : 2y\,(x+z) : 2z\,(x+z)\right] \\ &= [x:y:z]\,.\end{aligned}$$

This shows that $\alpha = \alpha\,(t:u)$. Note that our assumption on the field not having characteristic two is used here. If $x + z = 0$, then necessarily $y = 0$, so that $[x:y:z] = [-1:0:1]$. In this case $\alpha = \alpha\,(0:1)$.

To show uniqueness, suppose that we have two proportions $t:u$ and $r:s$ with

$$\left[t^2 - u^2 : 2tu : t^2 + u^2\right] = \left[r^2 - s^2 : 2rs : r^2 + s^2\right].$$

Then using the condition for equality of 3-proportions in (1),

$$\begin{aligned}\left(t^2 - u^2\right)(2rs) - \left(r^2 - s^2\right)(2tu) &= 2\,(ts - ru)\,(tr + us) = 0 \\ \left(t^2 - u^2\right)\left(r^2 + s^2\right) - \left(r^2 - s^2\right)\left(t^2 + u^2\right) &= 2\,(ts - ru)\,(ru + ts) = 0 \\ (2tu)\left(r^2 + s^2\right) - (2rs)\left(t^2 + u^2\right) &= 2\,(ts - ru)\,(us - tr) = 0.\end{aligned}$$

If $ts - ru \neq 0$ then it follows that

$$\begin{aligned}tr + us &= 0 \\ ru + st &= 0 \\ us - tr &= 0.\end{aligned}$$

Thus $tr = us = 0$ from the first and third equations, so at least one of $r, t$ must be zero and at least one of $s, u$ must be zero, but since $r:s$ and $t:u$ are both proportions, either $r = u = 0$ or $t = s = 0$, but not both. But this contradicts the second equation, so we conclude that $ts - ru = 0$, and the two proportions $t:u$ and $r:s$ are indeed equal. ∎

**Theorem 22 (Parametrization of null lines)** *Any null line $\Lambda$ is of the form*

$$\Lambda = \Lambda\,(t:u) \equiv \left(t^2 - u^2 : 2tu : t^2 + u^2\right)$$

*for some unique proportion $t:u$.*

**Proof.** This is dual to the Parametrization of null points theorem. ∎

It is worth pointing out that the three quadratic forms $t^2 - u^2, 2tu$ and $t^2 + u^2$ play a key role in the new theory of *chromogeometry* ([19]). The corresponding bilinear forms appear also in the next result.

**Theorem 23 (Join of null points)** *The join of the distinct null points* $\alpha_1 \equiv \alpha(t_1 : u_1)$ *and* $\alpha_2 \equiv \alpha(t_2 : u_2)$ *is the non-null line*

$$\alpha_1\alpha_2 = L(t_1 : u_1|t_2 : u_2) \equiv (t_1t_2 - u_1u_2 : t_1u_2 + t_2u_1 : t_1t_2 + u_1u_2).$$

**Proof.** Since

$$\begin{aligned}\alpha_1 &= \left[t_1^2 - u_1^2 : 2t_1u_1 : t_1^2 + u_1^2\right] \\ \alpha_2 &= \left[t_2^2 - u_2^2 : 2t_2u_2 : t_2^2 + u_2^2\right]\end{aligned}$$

the Join of points theorem shows that

$$\begin{aligned}\alpha_1\alpha_2 &= \begin{pmatrix} 2t_1u_1\left(t_2^2+u_2^2\right) - 2t_2u_2\left(t_1^2+u_1^2\right) : \left(t_1^2+u_1^2\right)\left(t_2^2-u_2^2\right) - \left(t_2^2+u_2^2\right)\left(t_1^2-u_1^2\right) \\ : 2t_1u_1\left(t_2^2-u_2^2\right) - 2t_2u_2\left(t_1^2-u_1^2\right)\end{pmatrix} \\ &= (2(t_2u_1 - t_1u_2)(t_1t_2 - u_1u_2) : 2(t_2u_1 - t_1u_2)(t_1u_2 + t_2u_1) : 2(t_2u_1 - t_1u_2)(t_1t_2 + u_1u_2)) \\ &= (t_1t_2 - u_1u_2 : t_1u_2 + t_2u_1 : t_1t_2 + u_1u_2).\end{aligned}$$

We have divided by the factor $2(t_2u_1 - t_1u_2)$, which is non-zero since the field does not have characteristic two, and since $t_1 : u_1$ and $t_2 : u_2$ are distinct proportions. The identity

$$(t_1t_2 - u_1u_2)^2 + (t_1u_2 + t_2u_1)^2 - (t_1t_2 + u_1u_2)^2 = (t_1u_2 - t_2u_1)^2. \tag{6}$$

shows that $L(t_1 : u_1|t_2 : u_2)$ is a non-null line. ■

**Theorem 24 (Meet of null lines)** *The meet of the distinct null lines* $\Lambda_1 \equiv \Lambda(t_1 : u_1)$ *and* $\Lambda_2 \equiv \Lambda(t_2 : u_2)$ *is the non-null point*

$$\Lambda_1\Lambda_2 = a(t_1 : u_1|t_2 : u_2) \equiv [t_1t_2 - u_1u_2 : t_1u_2 + t_2u_1 : t_1t_2 + u_1u_2].$$

**Proof.** This is dual to the Join of null points theorem. ■

**Theorem 25 (Null diagonal point)** *The meet of the disjoint lines* $L(t_1 : u_1|t_2 : u_2)$ *and* $L(t_3 : u_3|t_4 : u_4)$ *is the point*

$$a(t_1 : u_1|t_2 : u_2||t_3 : u_3|t_4 : u_4) \equiv [x : y : z]$$

*where*

$$\begin{aligned}x &\equiv (t_1u_2 + t_2u_1)(t_3t_4 + u_3u_4) - (t_3u_4 + t_4u_3)(t_1t_2 + u_1u_2) \\ y &\equiv (t_1t_2 + u_1u_2)(t_3t_4 - u_3u_4) - (t_3t_4 + u_3u_4)(t_1t_2 - u_1u_2) \\ z &\equiv (t_1u_2 + t_2u_1)(t_3t_4 - u_3u_4) - (t_3u_4 + t_4u_3)(t_1t_2 - u_1u_2).\end{aligned}$$

**Proof.** Finding the coordinates $x, y$ and $z$ is essentially the computation of

$$J(t_1t_2 - u_1u_2, t_1u_2 + t_2u_1, t_1t_2 + u_1u_2; t_3t_4 - u_3u_4, t_3u_4 + t_4u_3, t_3t_4 + u_3u_4). \quad ■$$

Note the pleasant identity

$$\begin{aligned}&((t_1u_2 + t_2u_1)(t_3t_4 + u_3u_4) - (t_3u_4 + t_4u_3)(t_1t_2 + u_1u_2))^2 \\ &+ ((t_1t_2 + u_1u_2)(t_3t_4 - u_3u_4) - (t_3t_4 + u_3u_4)(t_1t_2 - u_1u_2))^2 \\ &- ((t_1u_2 + t_2u_1)(t_3t_4 - u_3u_4) - (t_3u_4 + t_4u_3)(t_1t_2 - u_1u_2))^2 \\ &= 4(t_3u_2 - t_2u_3)(u_1t_3 - t_1u_3)(u_1t_4 - t_1u_4)(u_2t_4 - t_2u_4).\end{aligned}$$

**Theorem 26 (Null diagonal line)** *The join of the disjoint points* $a(t_1 : u_1|t_2 : u_2)$ *and* $a(t_3 : u_3|t_4 : u_4)$ *is the line*

$$L(t_1 : u_1|t_2 : u_2||t_3 : u_3|t_4 : u_4) \equiv (x : y : z)$$

*where* $x, y$ *and* $z$ *are as in the Null diagonal point theorem.*

**Proof.** This is dual to the Null diagonal point theorem. ■

**Theorem 27 (Perpendicular null line)** *If $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ are distinct points with $a_1$ null, then $a_1 a_2$ is a null line precisely when $a_1$ and $a_2$ are perpendicular, in which case $a_1 a_2 = a_1^{\perp}$.*

**Proof.** From the Join of points theorem

$$a_1 a_2 \equiv (y_1 z_2 - y_2 z_1 : z_1 x_2 - z_2 x_1 : x_2 y_1 - x_1 y_2).$$

The identity

$$\begin{aligned}&(y_1 z_2 - y_2 z_1)^2 + (z_1 x_2 - z_2 x_1)^2 - (x_2 y_1 - x_1 y_2)^2 \\ &= (x_1 x_2 + y_1 y_2 - z_1 z_2)^2 - \left(x_1^2 + y_1^2 - z_1^2\right)\left(x_2^2 + y_2^2 - z_2^2\right)\end{aligned}$$

shows that if $a_1$ is a null point, meaning that $x_1^2 + y_1^2 - z_1^2 = 0$, then $a_1 a_2$ is a null line precisely when $a_1$ and $a_2$ are perpendicular. In this case both $a_1$ and $a_2$ are perpendicular to $a_1$, so that $a_1 a_2 = a_1^{\perp}$. ■

**Theorem 28 (Perpendicular null point)** *If $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$ are distinct lines with $L_1$ null, then $L_1 L_2$ is a null point precisely when $L_1$ and $L_2$ are perpendicular, in which case $L_1 L_2 = L_1^{\perp}$.*

**Proof.** This is dual to the Perpendicular null line theorem. ■

**Theorem 29 (Parametrizing a null line)** *Suppose that $\Lambda = \Lambda(t : u)$ is a null line. If $t^2 + u^2 \neq 0$, then any point $a$ lying on $\Lambda$ is of the form*

$$\left[r\left(t^2 - u^2\right) - 2stu : 2rtu + s\left(t^2 - u^2\right) : r\left(t^2 + u^2\right)\right]$$

*for a unique proportion $r : s$, while if $t^2 + u^2 = 0$, then every point $a$ lying on $\Lambda$ is of the form*

$$[rt : ru : s]$$

*for a unique proportion $r : s$.*

**Proof.** Two points lying on $\Lambda(t : u) \equiv \left(t^2 - u^2 : 2tu : t^2 + u^2\right)$ are $\left[t^2 - u^2 : 2tu : t^2 + u^2\right]$ and $\left[-2tu : t^2 - u^2 : 0\right]$. If $t^2 + u^2 \neq 0$, these are distinct, and the Parametrization of a join theorem shows that for any point $a$ lying on $\Lambda$ there is a unique proportion $r : s$ such that

$$a = \left[r\left(t^2 - u^2\right) - 2stu : 2rtu + s\left(t^2 - u^2\right) : r\left(t^2 + u^2\right)\right].$$

On the other hand if $t^2 + u^2 = 0$ then $[t : u : 0]$ and $[0 : 0 : 1]$ are two distinct points lying on $\Lambda$, so again by the Parametrizing a join theorem, for any point $a$ lying on $\Lambda$ there is a unique proportion $r : s$ such that

$$a = [rt : ru : s]. \quad ■$$

**Theorem 30 (Parametrizing a null point)** *Suppose that $\alpha = \alpha(t : u)$ is a null point. If $t^2 + u^2 \neq 0$, then any line $L$ passing through $\alpha$ is of the form*

$$\left(r\left(t^2 - u^2\right) - 2stu : 2rtu + s\left(t^2 - u^2\right) : r\left(t^2 + u^2\right)\right)$$

*for a unique proportion $r : s$, while if $t^2 + u^2 = 0$, then every line $L$ passing through $\alpha$ is of the form*

$$(rt : ru : s)$$

*for a unique proportion $r : s$.*

**Proof.** This is dual to the Parametrizing a null line theorem. ■

## 2.10 Triangles and trilaterals

A **triangle** $\overline{a_1a_2a_3}$ is a set $\{a_1, a_2, a_3\}$ of three non-collinear points. A **trilateral** $\overline{L_1L_2L_3}$ is a set $\{L_1, L_2, L_3\}$ of three non-concurrent lines.

The triangle $\triangle \equiv \overline{a_1a_2a_3}$ has an **associated trilateral** $\widetilde{\triangle} \equiv \overline{L_1L_2L_3}$ consisting of the three **lines** of the triangle, namely

$$L_1 \equiv a_2a_3 \qquad L_2 \equiv a_1a_3 \qquad L_3 \equiv a_1a_2.$$

It also has a **dual trilateral** $\triangle^\perp \equiv \overline{a_1^\perp a_2^\perp a_3^\perp}$ consisting of the three **dual lines** of the triangle, namely $a_1^\perp$, $a_2^\perp$ and $a_3^\perp$.

A trilateral $\triangledown \equiv \overline{L_1L_2L_3}$ has an **associated triangle** $\widetilde{\triangledown} \equiv \overline{a_1a_2a_3}$ consisting of the three **points** of the trilateral, namely

$$a_1 \equiv L_2L_3 \qquad a_2 \equiv L_1L_3 \qquad a_3 \equiv L_1L_2.$$

It also has a **dual triangle** $\triangledown^\perp \equiv \overline{L_1^\perp L_2^\perp L_3^\perp}$ consisting of the three **dual points** of the trilateral, namely $L_1^\perp$, $L_2^\perp$ and $L_3^\perp$.

A triangle $\triangle \equiv \overline{a_1a_2a_3}$ and its associated trilateral $\widetilde{\triangle} \equiv \overline{L_1L_2L_3}$ both have **sides** $\overline{a_1a_2}$, $\overline{a_2a_3}$ and $\overline{a_1a_3}$, **vertices** $\overline{L_1L_2}$, $\overline{L_2L_3}$ and $\overline{L_1L_3}$, and **couples** $\overline{a_1L_1}$, $\overline{a_2L_2}$ and $\overline{a_3L_3}$.

**Theorem 31 (Triangle trilateral duality)** *If the triangle $\triangle \equiv \overline{a_1a_2a_3}$ is dual to the trilateral $\triangledown \equiv \overline{A_1A_2A_3}$, then the points, lines, sides, vertices and couples of $\overline{a_1a_2a_3}$ are dual respectively to the lines, points, vertices, sides and couples of $\overline{A_1A_2A_3}$.*

**Proof.** These statements all follow directly from the definitions. ∎

A triangle $\triangle \equiv \overline{a_1a_2a_3}$ is **null** precisely when one or more of its lines $a_1a_2, a_2a_3$ or $a_1a_3$ is null. More specifically, $\triangle$ is **singly-null** precisely when exactly one of its lines is null, **doubly-null** precisely when exactly two of its lines are null, and **triply-null** or **fully-null** precisely when all three of its lines are null.

**Example 5** *Given a proportion $t : u$, consider points $a_1 \equiv \left[t^2+u^2 : 0 : t^2-u^2\right]$, $a_2 \equiv \left[0 : t^2+u^2 : 2tu\right]$ and $a_3 \equiv \left[2tu : t^2-u^2 : 0\right]$. Then $\overline{a_1a_2a_3}$ is a fully-null triangle.* ⋄

A trilateral $\triangledown \equiv \overline{L_1L_2L_3}$ is **null** precisely when one or more of its points $L_1L_2, L_2L_3$ or $L_1L_3$ is null. More specifically, $\triangledown$ is **singly-null** precisely when exactly one of its points is null, **doubly-null** precisely when exactly two of its points are null, and **triply-null** or **fully-null** precisely when all three of its points are null.

A triangle $\triangle \equiv \overline{a_1a_2a_3}$ is **nil** precisely when one or more of its points $a_1, a_2$ or $a_3$ is null. More specifically, $\triangle$ is **singly-nil** precisely when exactly one of its points is null, **doubly-nil** precisely when exactly two of its points are null, and **triply-nil** or **fully-nil** precisely when all three of its points are null.

A trilateral $\triangledown \equiv \overline{L_1L_2L_3}$ is **nil** precisely when one or more of its lines $L_1, L_2$ or $L_3$ is null. More specifically, $\triangledown$ is **singly-nil** precisely when exactly one of its lines is null, **doubly-nil** precisely when exactly two of its lines are null, and **triply-nil** or **fully-nil** precisely when all three of its lines are null.

A triangle $\triangle \equiv \overline{a_1a_2a_3}$ is **right** precisely when one or more of its vertices is right. More specifically, $\triangle$ is **singly-right** precisely when it has exactly one right vertex, **doubly-right** precisely when it has exactly two right vertices, and **triply-right** or **fully-right** precisely when it has three right vertices.

A trilateral $\triangledown \equiv \overline{L_1L_2L_3}$ is **right** precisely when one or more of its sides is right. More specifically, $\triangledown$ is **singly-right** precisely when it has exactly one right side, **doubly-right** precisely when it has exactly two right sides, and **triply-right** or **fully-right** precisely when it has three right sides.

# 3 Hyperbolic trigonometry

In this section, which constitutes the heart of the paper, we introduce the basic measurements of universal hyperbolic geometry: the *quadrance* $q(a_1, a_2)$ between points $a_1$ and $a_2$, and the *spread* $S(L_1, L_2)$ between lines $L_1$ and $L_2$. These are *dual notions* in universal hyperbolic geometry. Recall that we work over a general field not of characteristic two: the values of the quadrance and spread will be *numbers in this field*.

After briefly describing how quadrance and spread may be interpreted using cross ratios and perpendicularity, we introduce the main trigonometric laws: the *Triple quad formula*, the *Triple spread formula*, *Pythagoras' theorem*, the *Spread law* and the *Cross law*, along with their duals. Along the way we also define the *quadrea* $\mathcal{A}$ of a triangle, and the *quadreal* $\mathcal{L}$ of a trilateral, and the secondary concepts of *product* and *cross* between points and lines respectively. Then we derive some variants of the main laws that assume the existence of *midpoints* and *midlines.*

The notions of quadrance and spread in this hyperbolic setting are closely related to the corresponding notions in planar Euclidean geometry, and the main laws can be seen as deformations of the Euclidean ones. In the special case of classical hyperbolic geometry in the Beltrami Klein model over the real numbers, the quadrance $q(a_1, a_2)$ between points $a_1$ and $a_2$ is related to the usual hyperbolic distance $d(a_1, a_2)$ between them by the relation

$$q(a_1, a_2) = -\sinh^2(d(a_1, a_2)). \tag{7}$$

The spread $S(L_1, L_2)$ between lines $L_1$ and $L_2$ is related to the usual hyperbolic angle $\theta(L_1, L_2)$ between them by the relation

$$S(L_1, L_2) = \sin^2(\theta(L_1, L_2)). \tag{8}$$

However our derivation of hyperbolic trigonometry is *completely independent of the classical treatment*, much more elementary, and extends far beyond it. So we can take any formula from universal hyperbolic geometry, make the substitutions (7) and (8), restrict to the interior of the null cone or unit disk, and obtain a valid formula for classical hyperbolic geometry over the real numbers.

### 3.1 Quadrance between points

The **quadrance** between points $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ is the number

$$q(a_1, a_2) \equiv 1 - \frac{(x_1x_2 + y_1y_2 - z_1z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)}. \tag{9}$$

This number is undefined if either of $a_1, a_2$ are null points, and has the value 1 precisely when $a_1$ and $a_2$ are perpendicular. If $a_1 = a_2$ then $q(a_1, a_2) = 0$, but we shall shortly see that there are also other ways in which a quadrance of zero can occur.

**Example 6** *The quadrance between* $a_1 \equiv [0 : 0 : 1]$ *and* $a \equiv [x : y : 1]$ *is*

$$q(a_1, a) = \frac{x^2 + y^2}{x^2 + y^2 - 1}. \quad \diamond$$

**Example 7** *The quadrance between* $a_2 \equiv [1 : 0 : 0]$ *and* $a \equiv [x : y : 1]$ *is*

$$q(a_2, a) = \frac{y^2 - 1}{x^2 + y^2 - 1}. \quad \diamond$$

**Example 8** *The quadrance between* $c_1 \equiv [x_1 : 0 : 1]$ *and* $c_2 \equiv [x_2 : 0 : 1]$ *is*

$$q(c_1, c_2) = \frac{-(x_1 - x_2)^2}{(x_2^2 - 1)(x_1^2 - 1)}. \quad \diamond$$

**Theorem 32 (Quadrance)** *For points* $a_1 \equiv [x_1 : y_1 : z_1]$ *and* $a_2 \equiv [x_2 : y_2 : z_2]$,

$$q(a_1, a_2) = -\frac{(y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 - (x_1y_2 - y_1x_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)}. \tag{10}$$

**Proof.** This is a consequence of the following extension of Fibonacci's identity:

$$\begin{aligned}&(x_2x_3 + y_2y_3 - z_2z_3)^2 - (y_2z_3 - y_3z_2)^2 - (z_2x_3 - z_3x_2)^2 + (x_3y_2 - x_2y_3)^2 \\ &= \left(x_2^2 + y_2^2 - z_2^2\right)\left(x_3^2 + y_3^2 - z_3^2\right). \quad \blacksquare\end{aligned}$$

*Note carefully the minus sign* that begins the right hand side of (10). This has the consequence that for points corresponding to classical hyperbolic geometry, *the quadrance between them is negative,* in agreement with (7).

**Theorem 33 (Zero quadrance)** *If* $a_1$ *and* $a_2$ *are distinct points, then* $q(a_1, a_2) = 0$ *precisely when* $a_1a_2$ *is a null line.*

**Proof.** This follows from the Quadrance theorem together with the Join of points theorem, since if $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ then

$$a_1a_2 \equiv (y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_2y_1 - x_1y_2)$$

and this is a null line precisely when

$$(y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 - (x_2y_1 - x_1y_2)^2 = 0. \quad \blacksquare$$

## 3.2 Spread between lines

The **spread** between lines $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$ is the number

$$S(L_1, L_2) \equiv 1 - \frac{(l_1 l_2 + m_1 m_2 - n_1 n_2)^2}{(l_1^2 + m_1^2 - n_1^2)(l_2^2 + m_2^2 - n_2^2)}. \tag{11}$$

This number is undefined if either of $L_1, L_2$ are null lines, and has the value 1 precisely when the lines $L_1$ and $L_2$ are perpendicular. If $L_1 = L_2$ then $S(L_1, L_2) = 0$, but again there are also other ways in which a spread of zero can occur.

**Theorem 34 (Spread)** *For lines $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$,*

$$S(L_1, L_2) = -\frac{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 - (l_1 m_2 - l_2 m_1)^2}{(l_1^2 + m_1^2 - n_1^2)(l_2^2 + m_2^2 - n_2^2)}. \tag{12}$$

**Proof.** This is dual to the Quadrance theorem. ■

**Example 9** *The spread between the non-null lines $L_1 \equiv (l_1 : m_1 : 0)$ and $L_2 \equiv (l_2 : m_2 : 0)$ passing through the point $[0 : 0 : 1]$ is*

$$S(L_1, L_2) = \frac{(l_1 m_2 - l_2 m_1)^2}{(l_1^2 + m_1^2)(l_2^2 + m_2^2)}.$$

*This is the same as the Euclidean planar spread (as studied in [17]) between the lines $\langle l_1 : m_1 : 0\rangle$ and $\langle l_2 : m_2 : 0\rangle$ with respective equations $l_1 X + m_1 Y = 0$ and $l_2 X + m_1 Y = 0$.* ⋄

**Example 10** *The spread between the non-null lines $M_1 \equiv (l_1 : 0 : 1)$ and $M_2 \equiv (l_2 : 0 : 1)$ passing through the point $[0 : 1 : 0]$ is*

$$S(M_1, M_2) = -\frac{(l_2 - l_1)^2}{(l_1^2 - 1)(l_2^2 - 1)}. \quad \diamond$$

**Theorem 35 (Zero spread)** *If $L_1$ and $L_2$ are distinct non-null lines, then $S(L_1, L_2) = 0$ precisely when $L_1 L_2$ is a null point.*

**Proof.** This is dual to the Zero quadrance theorem. ■

**Theorem 36 (Quadrance spread duality)** *If $a_1$ and $a_2$ are non-null points with $A_1 \equiv a_1^\perp$ and $A_2 \equiv a_2^\perp$ the dual lines, then*

$$q(a_1, a_2) = S(A_1, A_2).$$

**Proof.** This is obvious from the definitions. ■

## 3.3 Cross ratios

We now show that the quadrance between points and the spread between lines may be framed entirely within projective geometry with the additional notion of *perpendicularity* arising from the quadratic form $x^2 + y^2 - z^2$. The crucial link is provided by the *cross-ratio*. This naturally extends to other quadratic forms, see for example [20]. Since we are particularly interested here in explicit formulas in the hyperbolic setting, this generalization is not treated here, but nevertheless its existence should be kept in mind.

Note also that the following treatment relating quadrance to a certain cross ratio is *quite different* from the classical one over the real numbers expressing a hyperbolic distance $d(a_1, a_2)$ as one half of the log of a different cross-ratio—which requires the line $a_1 a_2$ to pass through two null points. Here that assumption is unnecessary. Furthermore our approach dualizes immediately to spreads between lines also.

To connect with projective geometry, we work in the vector space $\mathbb{F}^3$, with typical vector $v \equiv (a, b, c)$. Then $[v] = [a : b : c]$ is the associated (hyperbolic) point, as previously.

The cross-ratio may be viewed as an *affine quantity* that extends to be *projectively invariant*. Suppose that four non-zero vectors $v_1, v_2, u_1, u_2$ lie in a two-dimensional subspace spanned by vectors $p$ and $q$. Then we can write $v_1 = x_1 p + y_1 q$, $v_2 = x_2 p + y_2 q$, $u_1 = z_1 p + w_1 q$, and $u_2 = z_2 p + w_2 q$ uniquely. The **cross-ratio** is the ratio of ratios:

$$(v_1, v_2 : u_1, u_2) \equiv \frac{(x_1 w_1 - y_1 z_1)}{(x_1 w_2 - y_1 z_2)} \Big/ \frac{(x_2 w_1 - y_2 z_1)}{(x_2 w_2 - y_2 z_2)} = \frac{\begin{vmatrix} x_1 & y_1 \\ z_1 & w_1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 \\ z_2 & w_2 \end{vmatrix}} \Big/ \frac{\begin{vmatrix} x_2 & y_2 \\ z_1 & w_1 \end{vmatrix}}{\begin{vmatrix} x_2 & y_2 \\ z_2 & w_2 \end{vmatrix}}.$$

If we choose a new basis $p'$ and $q'$ with

$$\begin{aligned} p &= ap' + bq' \\ q &= cp' + dq' \end{aligned}$$

then the cross-ratio is unchanged, as each of the $2 \times 2$ determinants is multiplied by the determinant $ad - bc$ of the change of basis matrix. This ensures that the cross-ratio is indeed well-defined.

Furthermore the cross-ratio only depends on the central lines or (hyperbolic) points $[v_1], [v_2], [u_1]$ and $[u_2]$, since for example if we multiply $x_1$ and $y_1$ by a non-zero factor $\lambda$, then the cross-ratio is unchanged. So we write

$$([v_1], [v_2] : [u_1], [u_2]) \equiv (v_1, v_2 : u_1, u_2).$$

A useful identity is

$$([v_1], [v_2] : [u_1], [u_2]) + ([v_1], [u_1] : [v_2], [u_2]) = 1. \tag{13}$$

Now consider the special case when $p = v_1$ and $q = v_2$, so that $u_1 = z_1 v_1 + w_1 v_2$, and $u_2 = z_2 v_1 + w_2 v_2$. Then the cross ratio becomes simply

$$(v_1, v_2 : u_1, u_2) \equiv \frac{\begin{vmatrix} 1 & 0 \\ z_1 & w_1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ z_2 & w_2 \end{vmatrix}} \Big/ \frac{\begin{vmatrix} 0 & 1 \\ z_1 & w_1 \end{vmatrix}}{\begin{vmatrix} 0 & 1 \\ z_2 & w_2 \end{vmatrix}} = \frac{w_1}{w_2} \Big/ \frac{z_1}{z_2} = \frac{w_1 z_2}{w_2 z_1} = \frac{w_1}{z_1} \Big/ \frac{w_2}{z_2}. \tag{14}$$

**Theorem 37 (Quadrance cross ratio)** *Suppose that $\overline{a_1 a_2}$ is a non-null, non-nil side with opposite points*

$$o_1 \equiv (a_1 a_2)\, a_1^\perp \qquad \text{and} \qquad o_2 \equiv (a_1 a_2)\, a_2^\perp.$$

*Then*

$$q(a_1, a_2) = (a_1, o_2 : a_2, o_1).$$

**Proof.** Write $a_1 \equiv [v_1]$ and $a_2 \equiv [v_2]$ where $v_1 \equiv (x_1, y_1, z_1)$ and $v_2 \equiv (x_2, y_2, z_2)$. Then $v_1$ and $v_2$ are linearly independent, since $a_1$ and $a_2$ are distinct. As $\overline{a_1 a_2}$ is a non-null side, the Opposite points theorem states that the opposite points $o_1 = (a_1 a_2)\, a_1^\perp$ and $o_2 = (a_1 a_2)\, a_2^\perp$ can be expressed as

$$\begin{aligned} o_1 &= \left[ (x_1 x_2 + y_1 y_2 - z_1 z_2)\, v_1 - \left( x_1^2 + y_1^2 - z_1^2 \right) v_2 \right] \\ o_2 &= \left[ \left( x_2^2 + y_2^2 - z_2^2 \right) v_1 - (x_1 x_2 + y_1 y_2 - z_1 z_2)\, v_2 \right]. \end{aligned}$$

So $v_1, v_2, o_1$ and $o_2$ lie in the two-dimensional subspace spanned by $v_1$ and $v_2$. Using $v_1$ and $v_2$ as a basis, (14) shows that

$$\begin{aligned} (a_1, a_2 : o_2, o_1) &= \frac{-(x_1 x_2 + y_1 y_2 - z_1 z_2)}{(x_2^2 + y_2^2 - z_2^2)} \Big/ \frac{-\left( x_1^2 + y_1^2 - z_1^2 \right)}{(x_1 x_2 + y_1 y_2 - z_1 z_2)} \\ &= \frac{(x_1 x_2 + y_1 y_2 - z_1 z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)}. \end{aligned}$$

So using (13)

$$(a_1, o_2 : a_2, o_1) = 1 - (a_1, a_2 : o_2, o_1) = q(a_1, a_2). \quad \blacksquare$$

It is also possible to discuss cross-ratios of lines, and there is a dual result relating the spread between lines to such a cross-ratio.

### 3.4 Triple quad formula

We now come to the first two of the *main laws of hyperbolic trigonometry*: the *Triple quad formula* and the *Triple spread formula*. Both involve the **Triple spread function**

$$\begin{aligned} S(a,b,c) &\equiv (a+b+c)^2 - 2\left(a^2+b^2+c^2\right) - 4abc & (15)\\ &= 2ab+2ac+2bc-a^2-b^2-c^2-4abc & (16)\\ &= -\begin{vmatrix} 0 & a & b & 1\\ a & 0 & c & 1\\ b & c & 0 & 1\\ 1 & 1 & 1 & 2 \end{vmatrix} & (17)\\ &= 4(1-a)(1-b)(1-c) - (a+b+c-2)^2 & (18) \end{aligned}$$

which also plays a major role in planar rational trigonometry (see [17]). It should be noted that the Euclidean Triple quad formula

$$(Q_1+Q_2+Q_3)^2 = 2\left(Q_1^2+Q_2^2+Q_3^2\right) \tag{19}$$

is the relation between the three Euclidean quadrances formed by three collinear points. The hyperbolic version we now establish is a deformation of (19).

**Theorem 38 (Triple quad formula)** *If the points $a_1, a_2$ and $a_3$ are collinear, and $q_1 \equiv q(a_2,a_3)$, $q_2 \equiv q(a_1,a_3)$ and $q_3 \equiv q(a_1,a_2)$, then*

$$(q_1+q_2+q_3)^2 = 2\left(q_1^2+q_2^2+q_3^2\right) + 4q_1q_2q_3.$$

**Proof.** Suppose that the points are $a_1 \equiv [x_1 : y_1 : z_1]$, $a_2 \equiv [x_2 : y_2 : z_2]$ and $a_3 \equiv [x_3 : y_3 : z_3]$. Then

$$\begin{aligned} q_1 \equiv q(a_2,a_3) &= 1 - \frac{(x_2x_3+y_2y_3-z_2z_3)^2}{\left(x_2^2+y_2^2-z_2^2\right)\left(x_3^2+y_3^2-z_3^2\right)}\\ q_2 \equiv q(a_1,a_3) &= 1 - \frac{(x_1x_3+y_1y_3-z_1z_3)^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_3^2+y_3^2-z_3^2\right)}\\ q_3 \equiv q(a_1,a_2) &= 1 - \frac{(x_1x_2+y_1y_2-z_1z_2)^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_2^2+y_2^2-z_2^2\right)}. \end{aligned}$$

The expression

$$((1-q_1)+(1-q_2)+(1-q_3)-1)^2 - 4(1-q_1)(1-q_2)(1-q_3)$$

is a difference of squares. One of the factors is

$$\begin{aligned} &\frac{(x_2x_3+y_2y_3-z_2z_3)^2}{\left(x_2^2+y_2^2-z_2^2\right)\left(x_3^2+y_3^2-z_3^2\right)} + \frac{(x_1x_3+y_1y_3-z_1z_3)^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_3^2+y_3^2-z_3^2\right)} + \frac{(x_1x_2+y_1y_2-z_1z_2)^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_2^2+y_2^2-z_2^2\right)} - 1\\ &- \frac{2(x_2x_3+y_2y_3-z_2z_3)(x_1x_3+y_1y_3-z_1z_3)(x_1x_2+y_1y_2-z_1z_2)}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_2^2+y_2^2-z_2^2\right)\left(x_3^2+y_3^2-z_3^2\right)}. \end{aligned}$$

Somewhat remarkably, this expression is identically equally to

$$\frac{(x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_3y_2z_1 + x_3y_1z_2 - x_2y_1z_3)^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_2^2+y_2^2-z_2^2\right)\left(x_3^2+y_3^2-z_3^2\right)}. \tag{20}$$

If $a_1, a_2$ and $a_3$ are collinear, then from the Collinear points theorem the numerator of (20) is zero, so that

$$(2-q_1-q_2-q_3)^2 = 4(1-q_1)(1-q_2)(1-q_3). \tag{21}$$

Comparing (15) and (18), this can be rewritten as

$$(q_1+q_2+q_3)^2 = 2\left(q_1^2+q_2^2+q_3^2\right) + 4q_1q_2q_3. \quad \blacksquare$$

The Triple quad formula can be rewritten more compactly, using the Triple spread function, as

$$S(q_1,q_2,q_3) = 0.$$

**Example 11** *Consider the case when* $a_1 \equiv [0:0:1]$, $a_2 \equiv [x:y:1]$ *and* $a_3 \equiv [x:-y:1]$. *The corresponding lines are*

$$L_1 = (1:0:x) \qquad L_2 = (y:x:0) \qquad L_3 = (-y:x:0)\,.$$

*Then*

$$q_1 = \frac{4\left(x^2-1\right)y^2}{\left(x^2+y^2-1\right)^2} \qquad q_2 = \frac{x^2+y^2}{x^2+y^2-1} \qquad q_3 = \frac{x^2+y^2}{x^2+y^2-1}$$

*and a computation shows that*

$$S\left(q_1,q_2,q_3\right) = \frac{16\left(1+y^2\right)\left(1-x^2\right)x^2y^2}{\left(x^2+y^2-1\right)^4}.$$

*This is zero if* $x=0$ *or* $y=0$, *and also if* $x^2=1$, *in which case* $L_1 \equiv a_2a_3$ *is a null line, and if* $y \neq 0$ *then the three points* $a_1, a_2$ *and* $a_3$ *are not collinear. But there is another possibility—if we work in the field obtained by adjoining* $i$ *to the rational numbers, where* $i^2=-1$, *then setting* $y=i$ *we find an example involving non-null lines for which the converse of the Triple quad formula also does not hold.* $\diamond$

## 3.5 Triple spread formula

**Theorem 39 (Triple spread formula)** *If the lines* $L_1, L_2$ *and* $L_3$ *are concurrent, and* $S_1 \equiv S\left(L_2,L_3\right)$, $S_2 \equiv S\left(L_1,L_3\right)$ *and* $S_3 \equiv S\left(L_1,L_2\right)$, *then*

$$\left(S_1+S_2+S_3\right)^2 = 2\left(S_1^2+S_2^2+S_3^2\right)+4S_1S_2S_3.$$

**Proof.** This is dual to the Triple quad formula. ■

The Triple spread formula can also be rewritten as

$$S\left(S_1,S_2,S_3\right)=0.$$

Here are some secondary results that involve the Triple spread function, and apply to both collinear points and concurrent lines.

**Theorem 40 (Complementary quadrances spreads)** *Suppose that* $a, b$ *and* $c$ *are numbers which satisfy* $S\left(a,b,c\right)=0$. *If* $c=1$ *then*

$$a+b=1.$$

**Proof.** This is a consequence of the identity

$$S\left(a,b,c\right) = \left(1-c\right)\left(c+\left(1-2a\right)\left(1-2b\right)\right)-\left(a+b-1\right)^2. \ ■$$

The proof suggests that the converse of this result does not hold, since if $S\left(a,b,c\right)=0$ and $a+b=1$, then it is also possible that $c=-\left(1-2a\right)\left(1-2b\right)$. In fact we will see an example of just this situation in the Singly null singly nil theorem.

**Theorem 41 (Equal quadrances spreads)** *Suppose that* $a, b$ *and* $c$ *are numbers which satisfy* $S\left(a,b,c\right)=0$. *If* $a=b$ *then either*

$$c=0 \qquad \text{or} \qquad c=4a\left(1-a\right).$$

**Proof.** This is a consequence of the identity

$$S\left(a,b,c\right) = \left(a-b\right)\left(b-a-2c+4ac\right)-c\left(c-4a+4a^2\right). \ ■$$

The polynomial

$$S_2\left(x\right)=4x\left(1-x\right)$$

which appears in this theorem is the **second spread polynomial**. It plays a major role in hyperbolic geometry, and is also well-known in chaos theory as the *logistic map*.

### 3.6 Pythagoras' theorem

Pythagoras' theorem is of course very important. The planar Euclidean version involves the formula

$$Q_3 = Q_1 + Q_2.$$

The hyperbolic version involves a deformation of this.

**Theorem 42 (Pythagoras)** *Suppose that $a_1, a_2$ and $a_3$ are distinct points with quadrances $q_1 \equiv q(a_2, a_3)$, $q_2 \equiv q(a_1, a_3)$ and $q_3 \equiv q(a_1, a_2)$. If the lines $a_1a_3$ and $a_2a_3$ are perpendicular, then*

$$q_3 = q_1 + q_2 - q_1q_2.$$

**Proof.** Suppose that $a_1 \equiv [x_1 : y_1 : z_1]$, $a_2 \equiv [x_2 : y_2 : z_2]$ and $a_3 \equiv [x_3 : y_3 : z_3]$. The lines $a_1a_3$ and $a_2a_3$ are perpendicular precisely when

$$(y_2z_3 - y_3z_2)(y_3z_1 - y_1z_3) + (z_2x_3 - z_3x_2)(z_3x_1 - z_1x_3) - (x_3y_2 - x_2y_3)(x_1y_3 - x_3y_1) = 0.$$

This condition can be rewritten as

$$\left(x_3^2 + y_3^2 - z_3^2\right)(x_1x_2 + y_1y_2 - z_1z_2) - (x_2x_3 + y_2y_3 - z_2z_3)(x_1x_3 + y_1y_3 - z_1z_3) = 0. \tag{22}$$

Now from the definition of quadrance,

$$\begin{aligned}
&(1 - q_3) - (1 - q_1)(1 - q_2) \\
&= \frac{(x_1x_2 + y_1y_2 - z_1z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)} - \frac{(x_2x_3 + y_2y_3 - z_2z_3)^2}{(x_2^2 + y_2^2 - z_2^2)(x_3^2 + y_3^2 - z_3^2)} \frac{(x_1x_3 + y_1y_3 - z_1z_3)^2}{(x_1^2 + y_1^2 - z_1^2)(x_3^2 + y_3^2 - z_3^2)} \\
&= \frac{\left(x_3^2 + y_3^2 - z_3^2\right)^2 (x_1x_2 + y_1y_2 - z_1z_2)^2 - (x_2x_3 + y_2y_3 - z_2z_3)^2 (x_1x_3 + y_1y_3 - z_1z_3)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)(x_3^2 + y_3^2 - z_3^2)^2}.
\end{aligned} \tag{23}$$

The numerator is a difference of squares, and one of the factors is the left hand side of (22), so if $a_1a_3$ and $a_2a_3$ are perpendicular, then

$$1 - q_3 = (1 - q_1)(1 - q_2).$$

Now rewrite this as

$$q_3 = q_1 + q_2 - q_1q_2. \quad \blacksquare$$

Note that the converse does not follow from this argument, as the *other* factor of the numerator of (23) might be zero.

**Theorem 43 (Pythagoras' dual)** *Suppose that $L_1, L_2$ and $L_3$ are distinct lines with spreads $S_1 \equiv S(L_2, L_3)$, $S_2 \equiv S(L_1, L_3)$ and $S_3 \equiv S(L_1, L_2)$. If the points $L_1L_3$ and $L_2L_3$ are perpendicular, then*

$$S_3 = S_1 + S_2 - S_1S_2.$$

**Proof.** This is dual to Pythagoras' theorem. $\blacksquare$

**Example 12** *Consider the triangle $\overline{a_1a_2a_3}$ where $a_1 \equiv [x : 0 : 1]$, $a_2 \equiv [0 : y : 1]$ and $a_3 \equiv [0 : 0 : 1]$ as in Example 3. The lines are*

$$L_1 \equiv a_2a_3 = (1 : 0 : 0) \qquad L_2 \equiv a_1a_3 = (0 : 1 : 0) \qquad L_3 \equiv a_1a_2 = (y : x : xy).$$

*The quadrances are*

$$q_1 = -\frac{y^2}{1 - y^2} \qquad q_2 = -\frac{x^2}{1 - x^2} \qquad q_3 = \frac{x^2y^2 - x^2 - y^2}{(1 - x^2)(1 - y^2)}$$

*and the spreads are*

$$S_1 = \frac{\left(1 - x^2\right)y^2}{x^2 + y^2 - x^2y^2} \qquad S_2 = \frac{x^2\left(1 - y^2\right)}{x^2 + y^2 - x^2y^2} \qquad S_3 = 1.$$

*Since $S_3 = 1$, the lines $L_1$ and $L_2$ are perpendicular, and Pythagoras' theorem asserts that $q_3 = q_1 + q_2 - q_1q_2$, which can be verified directly.* $\diamond$

**Example 13** *Consider the triangle $\overline{a_1a_2a_3}$ where $a_1 \equiv [0:0:1]$, $a_2 \equiv [x:y:1]$ and $a_3 \equiv [x:-y:1]$ as in Example 11. The spread $S_1 \equiv S(a_1a_2, a_1a_3)$ is*

$$S_1 = \frac{4x^2y^2}{(x^2+y^2)^2}$$

*Using the quadrances computed in Example 11,*

$$q_1 - q_2 - q_3 + q_2q_3 = -\frac{\left(x^2-y^2-2\right)\left(x^2-y^2\right)}{(x^2+y^2-1)^2}$$

*while*

$$1 - S_1 = 1 - \frac{4x^2y^2}{(x^2+y^2)^2} = \frac{\left(x^2-y^2\right)^2}{(x^2+y^2)^2}.$$

*So we see that $S_1 = 1$ does imply $q_1 = q_2 + q_3 - q_1q_2$, verifying again Pythagoras' theorem. But if $x^2 - y^2 = 2$ and $x \neq \pm y$, then $q_1 = q_2 + q_3 - q_1q_2$ while $S_1 \neq 1$. This shows that the converse of Pythagoras' theorem does not hold.* $\diamond$

## 3.7 Product and cross

In planar rational trigonometry, if $s \equiv s(l_1, l_2)$ is the spread between two lines $l_1$ and $l_2$, then $c(l_1, l_2) \equiv 1 - s$ is called the *cross* between the lines. This is a useful concept in its own right, and here we consider the hyperbolic version, together with the corresponding notion between points, called *product*. The latter has no analogue in Euclidean geometry. Here are the definitions.

The **product** between points $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ is the number

$$p(a_1, a_2) \equiv \frac{(x_1x_2 + y_1y_2 - z_1z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)}.$$

The **cross** between lines $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$ is the number

$$C(L_1, L_2) \equiv \frac{(l_1l_2 + m_1m_2 - n_1n_2)^2}{(l_1^2 + m_1^2 - n_1^2)(l_2^2 + m_2^2 - n_2^2)}.$$

Then clearly

$$q(a_1, a_2) + p(a_1, a_2) = 1$$

and

$$S(L_1, L_2) + C(L_1, L_2) = 1.$$

Pythagoras' theorem can be rewritten in terms of products as

$$p_3 = p_1p_2$$

while Pythagoras' dual theorem can be rewritten as

$$C_3 = C_1C_2.$$

The Triple quad formula can be rewritten as

$$(p_1 + p_2 + p_3 - 1)^2 = 4p_1p_2p_3$$

while the Triple spread formula can be rewritten as

$$(C_1 + C_2 + C_3 - 1)^2 = 4C_1C_2C_3.$$

## 3.8 The Spread law

The Spread law is the analog in universal hyperbolic geometry of the *Sine law*. First we establish a useful, but somewhat lengthy, formula.

**Theorem 44 (Spread formula)** *Suppose that $a_1 \equiv [x_1 : y_1 : z_1]$, $a_2 \equiv [x_2 : y_2 : z_2]$ and $a_3 \equiv [x_3 : y_3 : z_3]$ are distinct points with $a_1a_2$ and $a_1a_3$ both non-null lines. Then $S_1 \equiv S(a_1a_2, a_1a_3)$ is given by the expression*

$$S_1 = -\frac{\left(x_1^2 + y_1^2 - z_1^2\right)\left(x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_3y_2z_1 + x_3y_1z_2 - x_2y_1z_3\right)^2}{\left(l_2^2 + m_2^2 - n_2^2\right)\left(l_3^2 + m_3^2 - n_3^2\right)} \tag{24}$$

*where*

$$\begin{aligned} l_2 &\equiv y_3z_1 - y_1z_3 & m_2 &\equiv z_3x_1 - z_1x_3 & n_2 &\equiv x_1y_3 - x_3y_1 \\ l_3 &\equiv y_1z_2 - y_2z_1 & m_3 &\equiv z_1x_2 - z_2x_1 & n_3 &\equiv x_2y_1 - x_1y_2. \end{aligned}$$

**Proof.** By the Join of points theorem

$$\begin{aligned} a_1a_3 &= (l_2 : m_2 : n_2) \\ a_1a_2 &= (l_3 : m_3 : n_3) \end{aligned}$$

where $l_2, m_2, n_2, l_3, m_3$ and $n_3$ are as in the statement of the theorem.

From the Spread theorem

$$S_1 = -\frac{(m_2n_3 - m_3n_2)^2 + (n_2l_3 - n_3l_2)^2 - (l_2m_3 - l_3m_2)^2}{\left(l_2^2 + m_2^2 - n_2^2\right)\left(l_3^2 + m_3^2 - n_3^2\right)}. \tag{25}$$

Now the polynomial identity

$$\begin{aligned} &\left((z_3x_1 - z_1x_3)(x_2y_1 - x_1y_2) - (z_1x_2 - z_2x_1)(x_1y_3 - x_3y_1)\right)^2 \\ &+ \left((x_1y_3 - x_3y_1)(y_1z_2 - y_2z_1) - (x_2y_1 - x_1y_2)(y_3z_1 - y_1z_3)\right)^2 \\ &- \left((y_3z_1 - y_1z_3)(z_1x_2 - z_2x_1) - (y_1z_2 - y_2z_1)(z_3x_1 - z_1x_3)\right)^2 \\ &= \left(x_1^2 + y_1^2 - z_1^2\right)\left(x_1y_3z_2 - x_1y_2z_3 + x_2y_1z_3 - x_2z_1y_3 - y_1x_3z_2 + x_3y_2z_1\right)^2 \end{aligned}$$

applied to the numerator of the right hand side of (25) gives

$$S_1 = -\frac{\left(x_1^2 + y_1^2 - z_1^2\right)\left(x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_3y_2z_1 + x_3y_1z_2 - x_2y_1z_3\right)^2}{\left(l_2^2 + m_2^2 - n_2^2\right)\left(l_3^2 + m_3^2 - n_3^2\right)}. \quad \blacksquare \tag{26}$$

The hyperbolic Spread law is essentially the same as the Euclidean one, which has the form

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}.$$

**Theorem 45 (Spread law)** *Suppose that $a_1, a_2$ and $a_3$ are distinct points with quadrances $q_1 \equiv q(a_2, a_3)$, $q_2 \equiv q(a_1, a_3)$ and $q_3 \equiv q(a_1, a_2)$, and spreads $S_1 \equiv S(a_1a_2, a_1a_3)$, $S_2 \equiv S(a_1a_2, a_2a_3)$ and $S_3 \equiv S(a_1a_3, a_2a_3)$. Then*

$$\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}.$$

**Proof.** If the spreads $S_1, S_2$ and $S_3$ are defined, then each of the lines $a_1a_2$, $a_2a_3$ and $a_1a_3$ is non-null and so by the Zero quadrance theorem each of the quadrances $q_1, q_2$ and $q_3$ is non-zero. Using the notation of the previous theorem, as well as

$$l_1 \equiv y_2z_3 - y_3z_2 \qquad m_1 \equiv z_2x_3 - z_3x_2 \qquad n_1 \equiv x_3y_2 - x_2y_3$$

the Quadrance theorem gives

$$q_1 = -\frac{l_1^2 + m_1^2 - n_1^2}{\left(x_2^2 + y_2^2 - z_2^2\right)\left(x_3^2 + y_3^2 - z_3^2\right)}. \tag{27}$$

Combine the Spread formula (24) with (27) to get

$$\begin{aligned}\frac{S_1}{q_1} &= \frac{\left(x_1^2+y_1^2-z_1^2\right)\left(x_2^2+y_2^2-z_2^2\right)\left(x_3^2+y_3^2-z_3^2\right)}{\left(l_1^2+m_1^2-n_1^2\right)\left(l_2^2+m_2^2-n_2^2\right)\left(l_3^2+m_3^2-n_3^2\right)} \\ &\quad\times\left(x_1y_2z_3-x_1y_3z_2+x_2y_3z_1-x_3y_2z_1+x_3y_1z_2-x_2y_1z_3\right)^2. \end{aligned}\tag{28}$$

Since this a *symmetric* rational expression in the three indices, we conclude that

$$\frac{S_1}{q_1}=\frac{S_2}{q_2}=\frac{S_3}{q_3}. \quad \blacksquare$$

The dual of the Spread law is essentially the same, but to maintain the duality principle we state it separately.

**Theorem 46 (Spread dual law)** *Suppose that $L_1, L_2$ and $L_3$ are distinct lines with spreads $S_1 \equiv S(L_2, L_3)$, $S_2 \equiv S(L_1, L_3)$ and $S_3 \equiv S(L_1, L_2)$, and quadrances $q_1 \equiv q(L_1L_2, L_1L_3)$, $q_2 \equiv q(L_1L_2, L_2L_3)$ and $q_3 \equiv q(L_1L_3, L_2L_3)$. Then*

$$\frac{q_1}{S_1}=\frac{q_2}{S_2}=\frac{q_3}{S_3}.$$

**Proof.** This is dual to the Spread law. $\blacksquare$

## 3.9 Quadrea and quadreal

If $a_1 \equiv [x_1 : y_1 : z_1]$, $a_2 \equiv [x_2 : y_2 : z_2]$ and $a_3 \equiv [x_3 : y_3 : z_3]$ are non-null points, then the **quadrea** of $\{a_1, a_2, a_3\}$ is the number

$$\mathcal{A} \equiv \mathcal{A}(a_1,a_2,a_3) \equiv -\frac{\left(x_1y_2z_3-x_1y_3z_2+x_2y_3z_1-x_3y_2z_1+x_3y_1z_2-x_2y_1z_3\right)^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_2^2+y_2^2-z_2^2\right)\left(x_3^2+y_3^2-z_3^2\right)}.\tag{29}$$

This is well-defined, and if the points are non-collinear and so form a triangle $\overline{a_1a_2a_3}$, then we write $\mathcal{A} \equiv \mathcal{A}\left(\overline{a_1a_2a_3}\right)$. Note the minus sign. The quadrea defined here in this essentially projective setting and the one defined in the planar situation of [17] are analogous, with the latter being 16 times the square of the area of the triangle.

**Theorem 47 (Quadrea)** *Suppose that $a_1, a_2$ and $a_3$ are distinct points with quadrances $q_2 \equiv q(a_1, a_3)$ and $q_3 \equiv q(a_1, a_2)$, spread $S_1 \equiv S(a_1a_2, a_1a_3)$ and quadrea $\mathcal{A}$. Then*

$$q_2q_3S_1 = \mathcal{A}.$$

**Proof.** Since the two quadrances $q_2$ and $q_3$ are defined, the points $a_1, a_2$ and $a_3$ are non-null. Using the notation of the Spread formula,

$$\begin{aligned} q_2 &= -\frac{l_2^2+m_2^2-n_2^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_3^2+y_3^2-z_3^2\right)} \\ q_3 &= -\frac{l_3^2+m_3^2-n_3^2}{\left(x_1^2+y_1^2-z_1^2\right)\left(x_2^2+y_2^2-z_2^2\right)}. \end{aligned}$$

Combining this with the Spread formula gives

$$q_2q_3S_1 = \mathcal{A}. \quad \blacksquare$$

This argument works even if $q_1 = q(a_2, a_3) = 0$, in which case $S_2$ and $S_3$ are not defined. In case all three quadrances $q_1, q_2$ and $q_3$ are non-zero, and all three spreads $S_1, S_2$ and $S_3$ are defined, symmetry gives

$$q_2q_3S_1 = q_1q_3S_2 = q_1q_2S_3 = \mathcal{A}.$$

It follows that we can rewrite the Spread law as

$$\frac{S_1}{q_1}=\frac{S_2}{q_2}=\frac{S_3}{q_3}=\frac{\mathcal{A}}{q_1q_2q_3}.$$

To compare, the Quadrea theorem in the planar situation has the form

$$\mathcal{A} = 4q_2q_3S_1.$$

If $L_1 \equiv (l_1 : m_1 : n_1)$, $L_2 \equiv (l_2 : m_2 : n_2)$ and $L_3 \equiv (l_3 : m_3 : n_3)$ are non-null lines, then the **quadreal** of $\{L_1, L_2, L_3\}$ is the number

$$\mathcal{L} \equiv \mathcal{L}(L_1, L_2, L_3) \equiv -\frac{(l_1m_2n_3 - l_1m_3n_2 + l_2m_3n_1 - l_3m_2n_1 + l_3m_1n_2 - l_2m_1n_3)^2}{(l_1^2 + m_1^2 - n_1^2)(l_2^2 + m_2^2 - n_2^2)(l_3^2 + m_3^2 - n_3^2)}. \tag{30}$$

This is well-defined, and if the lines are non-concurrent and so form a trilateral $\overline{L_1L_2L_3}$, then we write $\mathcal{L} = \mathcal{L}\left(\overline{L_1L_2L_3}\right)$.

**Theorem 48 (Quadreal)** *Suppose that $L_1, L_2$ and $L_3$ are distinct lines with spreads $S_2 \equiv S(L_1, L_3)$ and $S_3 \equiv S(L_1, L_2)$, quadrance $q_1 \equiv q(L_1L_2, L_1L_3)$ and quadreal $\mathcal{L}$. Then*

$$S_2S_3q_1 = \mathcal{L}.$$

**Proof.** This is dual to the Quadrea theorem. ■

It is convenient to also define the **quadreal** $\mathcal{L} \equiv \mathcal{L}(\overline{a_1a_2a_3})$ **of a triangle** to be the quadreal of its associated trilateral, and the **quadrea** $\mathcal{A} = \mathcal{A}\left(\overline{L_1L_2L_3}\right)$ **of a trilateral** to be the quadrea of its associated triangle.

**Theorem 49 (Quadrea quadreal product)** *Suppose that $a_1, a_2$ and $a_3$ are distinct points with quadrances $q_1 \equiv q(a_2, a_3)$, $q_2 \equiv q(a_1, a_3)$ and $q_3 \equiv q(a_1, a_2)$, spreads $S_1 \equiv S(a_1a_2, a_1a_3)$, $S_2 \equiv S(a_1a_2, a_2a_3)$ and $S_3 \equiv S(a_1a_3, a_2a_3)$, quadrea $\mathcal{A} \equiv \mathcal{A}(a_1, a_2, a_3)$ and quadreal $\mathcal{L} \equiv \mathcal{L}(L_1, L_2, L_3)$. Then*

$$\mathcal{A}\mathcal{L} = q_1q_2q_3S_1S_2S_3.$$

**Proof.** Combine the Quadrea theorem and the Quadreal theorem. ■

**Example 14** *Consider the case when $a_1 \equiv [0 : 0 : 1]$, $a_2 \equiv [x : y : 1]$ and $a_3 \equiv [x : -y : 1]$ as in Example 12, with quadrances*

$$q_1 = \frac{4\left(x^2 - 1\right)y^2}{\left(x^2 + y^2 - 1\right)^2} \qquad q_2 = \frac{x^2 + y^2}{x^2 + y^2 - 1} \qquad q_3 = \frac{x^2 + y^2}{x^2 + y^2 - 1}$$

*and spreads*

$$S_1 = \frac{4x^2y^2}{\left(x^2 + y^2\right)^2} \qquad S_2 = \frac{\left(x^2 + y^2 - 1\right)x^2}{\left(x^2 - 1\right)\left(x^2 + y^2\right)} \qquad S_3 = \frac{\left(x^2 + y^2 - 1\right)x^2}{\left(x^2 - 1\right)\left(x^2 + y^2\right)}.$$

*The quadrea and quadreal are*

$$\mathcal{A} = \frac{4x^2y^2}{\left(x^2 + y^2 - 1\right)^2} \qquad \text{and} \qquad \mathcal{L} = \frac{4x^4y^2}{\left(x^2 - 1\right)\left(x^2 + y^2\right)^2}$$

*and you may verify the Quadrea quadreal product theorem directly.* ⋄

## 3.10 The Cross law

The Cross law is the analog of the *Cosine law*. In planar rational trigonometry the Cross law has the form

$$\left(Q_1 - Q_2 - Q_3\right)^2 = 4Q_2Q_3\left(1 - s_1\right) \tag{31}$$

involving three quadrances and one spread, and it is hard to overstate the importance of this most powerful formula. In the hyperbolic setting, the Cross law is more complicated, but still very fundamental.

**Theorem 50 (Cross law)** *Suppose that $a_1, a_2$ and $a_3$ are distinct points with quadrances $q_1 \equiv q(a_2, a_3)$, $q_2 \equiv q(a_1, a_3)$ and $q_3 \equiv q(a_1, a_2)$, and spread $S_1 \equiv S(a_1a_2, a_1a_3)$. Then*

$$\left(q_2q_3S_1 - q_1 - q_2 - q_3 + 2\right)^2 = 4\left(1 - q_1\right)\left(1 - q_2\right)\left(1 - q_3\right). \tag{32}$$

**Proof.** Suppose that $a_1 \equiv [x_1 : y_1 : z_1]$, $a_2 \equiv [x_2 : y_2 : z_2]$ and $a_3 \equiv [x_3 : y_3 : z_3]$. The assumption that all three quadrances are defined implies that the three points are non-null. Square both sides of the polynomial identity

$$\begin{aligned}
&-(x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_3y_2z_1 + x_3y_1z_2 - x_2y_1z_3)^2 \\
&+ \left(x_1^2 + y_1^2 - z_1^2\right)(x_2x_3 + y_2y_3 - z_2z_3)^2 + \left(x_2^2 + y_2^2 - z_2^2\right)(x_1x_3 + y_1y_3 - z_1z_3)^2 \\
&+ \left(x_3^2 + y_3^2 - z_3^2\right)(x_1x_2 + y_1y_2 - z_1z_2)^2 \\
&- \left(x_1^2 + y_1^2 - z_1^2\right)\left(x_2^2 + y_2^2 - z_2^2\right)\left(x_3^2 + y_3^2 - z_3^2\right) \\
&= 2(x_2x_3 + y_2y_3 - z_2z_3)(x_1x_3 + y_1y_3 - z_1z_3)(x_1x_2 + y_1y_2 - z_1z_2)
\end{aligned} \tag{33}$$

and divide by

$$\left(x_1^2 + y_1^2 - z_1^2\right)^2 \left(x_2^2 + y_2^2 - z_2^2\right)^2 \left(x_3^2 + y_3^2 - z_3^2\right)^2$$

to deduce that if $\mathcal{A} \equiv \mathcal{A}(a_1, a_2, a_3)$ then

$$(\mathcal{A} + (1 - q_1) + (1 - q_2) + (1 - q_3) - 1)^2 = 4(1 - q_1)(1 - q_2)(1 - q_3).$$

Rewrite this as

$$(\mathcal{A} - q_1 - q_2 - q_3 + 2)^2 = 4(1 - q_1)(1 - q_2)(1 - q_3) \tag{34}$$

and use the Quadrea theorem to replace $\mathcal{A}$ by $q_2q_3S_1$. ∎

The Cross law gives a quadratic equation for the spreads of a triangle given the quadrances. So the three quadrances of a triangle do not quite determine its spreads. As a quadratic equation in $\mathcal{A}$, (34) can be rewritten using the Triple spread function as

$$\mathcal{A}^2 - 2(q_1 + q_2 + q_3 - 2)\mathcal{A} = S(q_1, q_2, q_3).$$

Motivated by the Cross law, we define the **Cross function**

$$C(\mathcal{A}, q_1, q_2, q_3) \equiv (\mathcal{A} - q_1 - q_2 - q_3 + 2)^2 - 4(1 - q_1)(1 - q_2)(1 - q_3). \tag{35}$$

**Example 15** *Suppose that a triangle $\overline{a_1a_2a_3}$ has equal quadrances $q_1 = q_2 = q_3 \equiv -3$. Then*

$$C(\mathcal{A}, -3, -3, -3) = (\mathcal{A} + 11)^2 - 256 = 0$$

*has solutions $\mathcal{A} = -27$ and $\mathcal{A} = 5$, and from the Quadrea theorem we deduce that*

$$S_1 = S_2 = S_3 = -3 \qquad or \qquad S_1 = S_2 = S_3 = \frac{5}{9}.$$

*Two triangles $\overline{a_1a_2a_3}$ that have these quadrances and spreads can be found over the respective fields $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$ and $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}, \sqrt{5}\right)$, with*

$$a_1 \equiv \left[\sqrt{2} : 0 : 1\right] \qquad a_2 \equiv \left[-1 : \sqrt{3} : \sqrt{2}\right] \qquad a_3 \equiv \left[-1 : -\sqrt{3} : \sqrt{2}\right]$$

*and*

$$a_1 \equiv \left[\sqrt{2} : 0 : \sqrt{5}\right] \qquad a_2 \equiv \left[-1 : \sqrt{3} : \sqrt{10}\right] \qquad a_3 \equiv \left[-1 : -\sqrt{3} : \sqrt{10}\right]. \ \diamond$$

It is an instructive exercise to verify that both of the classical hyperbolic Cosine laws

$$\cosh d_1 = \cosh d_2 \cosh d_3 - \sinh d_2 \sinh d_3 \cos\theta_1 \tag{36}$$

and

$$\cosh d_1 = \frac{\cos\theta_2\cos\theta_3 + \cos\theta_1}{\sin\theta_2\sin\theta_3} \tag{37}$$

relating lengths $d_1, d_2, d_3$ and angles $\theta_1, \theta_2, \theta_3$ in a classical hyperbolic triangle can be manipulated using (7) and (8) to obtain the Cross law.

**Theorem 51 (Cross dual law)** *Suppose that $L_1, L_2$ and $L_3$ are distinct lines with spreads $S_1 \equiv S(L_2, L_3)$, $S_2 \equiv S(L_1, L_3)$ and $S_3 \equiv S(L_1, L_2)$, and quadrance $q_1 \equiv q(L_1L_2, L_1L_3)$. Then*

$$(S_2S_3q_1 - S_1 - S_2 - S_3 + 2)^2 = 4(1 - S_1)(1 - S_2)(1 - S_3).$$

**Proof.** This is dual to the Cross law. ∎

This can also be restated in terms of the quadreal $\mathcal{L} \equiv \mathcal{L}(L_1, L_2, L_3)$ as

$$(\mathcal{L} - S_1 - S_2 - S_3 + 2)^2 = 4(1 - S_1)(1 - S_2)(1 - S_3).$$

### 3.11 Alternate formulations

As in the Euclidean case, the most powerful of the trigonometric laws is the Cross law

$$(q_2 q_3 S_1 - q_1 - q_2 - q_3 + 2)^2 = 4(1-q_1)(1-q_2)(1-q_3). \tag{38}$$

In the special case $S_1 = 0$, we get

$$(q_1 + q_2 + q_3 - 2)^2 = 4(1-q_1)(1-q_2)(1-q_3)$$

which is equivalent to the Triple quad formula

$$(q_1 + q_2 + q_3)^2 = 2\left(q_1^2 + q_2^2 + q_3^2\right) + 4q_1 q_2 q_3.$$

If we rewrite (38) in the form

$$(q_1 - q_2 - q_3 + q_2 q_3 S_1)^2 = 4q_2 q_3 (1-q_1)(1-S_1) \tag{39}$$

then in the special case $S_1 = 1$ we recover Pythagoras' theorem in the form

$$q_1 = q_2 + q_3 - q_2 q_3.$$

Also (39) may be viewed as a deformation of the planar Cross law (31).

### 3.12 Triple product and triple cross

There are also secondary invariants associated to three points or three lines, besides the quadrea and the quadreal. The **triple product** of the three non-null points $a_1, a_2$ and $a_3$ is the number

$$\mathcal{P} \equiv \mathcal{P}(a_1, a_2, a_3) \equiv \frac{(x_2x_3 + y_2y_3 - z_2z_3)(x_1x_3 + y_1y_3 - z_1z_3)(x_1x_2 + y_1y_2 - z_1z_2)}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)(x_3^2 + y_3^2 - z_3^2)}.$$

The **triple cross** of the three non-null lines $L_1, L_2$ and $L_3$ is the number

$$\mathcal{C} = \mathcal{C}(L_1, L_2, L_3) \equiv \frac{(l_2l_3 + m_2m_3 - n_2n_3)(l_1l_3 + m_1m_3 - n_1n_3)(l_1l_2 + m_1m_2 - n_1n_2)}{(l_1^2 + m_1^2 - n_1^2)(l_2^2 + m_2^2 - n_2^2)(l_3^2 + m_3^2 - n_3^2)}.$$

**Theorem 52 (Triple product relation)** *Suppose that the three points $a_1, a_2, a_3$ have quadrea $\mathcal{A} \equiv \mathcal{A}(a_1, a_2, a_3)$, triple product $\mathcal{P} \equiv \mathcal{P}(a_1, a_2, a_3)$, and products $p_1 \equiv p(a_2, a_3)$, $p_2 \equiv p(a_1, a_3)$ and $p_3 \equiv p(a_1, a_2)$. Then*

$$\mathcal{A} + p_1 + p_2 + p_3 - 1 = 2\mathcal{P}.$$

**Proof.** This is a consequence of the algebraic identity (33). ■

**Theorem 53 (Triple cross relation)** *Suppose that the three lines $L_1, L_2, L_3$ have quadreal $\mathcal{L} \equiv \mathcal{L}(L_1, L_2, L_3)$, triple cross $\mathcal{C} \equiv \mathcal{C}(L_1, L_2, L_3)$, and crosses $C_1 \equiv C(L_2, L_3)$, $C_2 \equiv C(L_1, L_3)$ and $C_3 \equiv C(L_1, L_2)$. Then*

$$\mathcal{L} + C_1 + C_2 + C_3 - 1 = 2\mathcal{C}.$$

**Proof.** This is dual to the Triple product relation. ■

The Cross law in the form (34) may be seen to be the result of squaring the Triple product relation and replacing $\mathcal{P}^2$ with $p_1p_2p_3$, and similarly for the Cross dual law.

### 3.13 Midpoints and midlines

A **midpoint** of the side $\overline{a_1a_2}$ is a point $m$ which lies on $a_1a_2$ and satisfies

$$q(a_1, m) = q(a_2, m). \tag{40}$$

**Theorem 54 (Midpoints)** *The side $\overline{a_1 a_2}$ has a midpoint precisely when $p(a_1, a_2) = 1 - q(a_1, a_2)$ is a square. In this case if $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$, then we can renormalize so that*

$$x_1^2 + y_1^2 - z_1^2 = x_2^2 + y_2^2 - z_2^2$$

*and then there are exactly two midpoints, namely*

$$m_1 \equiv [x_1 + x_2 : y_1 + y_2 : z_1 + z_2] \qquad \text{and} \qquad m_2 \equiv [x_1 - x_2 : y_1 - y_2 : z_1 - z_2].$$

*Furthermore $m_1 \perp m_2$.*

**Proof.** If $a_1 \equiv [x_1 : y_1 : z_1]$, $a_2 \equiv [x_2 : y_2 : z_2]$ and $m \equiv [x : y : z]$, then (40) becomes

$$1 - \frac{(x_1 x + y_1 y - z_1 z)^2}{(x_1^2 + y_1^2 - z_1^2)(x^2 + y^2 - z^2)} = 1 - \frac{(x_2 x + y_2 y - z_2 z)^2}{(x_2^2 + y_2^2 - z_2^2)(x^2 + y^2 - z^2)}.$$

In order for this to have a solution, $(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)$ must be a square, or equivalently

$$p(a_1, a_2) \equiv \frac{(x_1 x_2 + y_1 y_2 - z_1 z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)} = 1 - q(a_1, a_2)$$

must be a square. In this case, after renormalization we may assume that

$$x_1^2 + y_1^2 - z_1^2 = x_2^2 + y_2^2 - z_2^2.$$

Then $q(a_1, m) = q(a_2, m)$ is equivalent to

$$\begin{aligned} 0 &= (x_1 x + y_1 y - z_1 z)^2 - (x_2 x + y_2 y - z_2 z)^2 \\ &= (x(x_1 - x_2) + y(y_1 - y_2) - z(z_1 - z_2))(x(x_1 + x_2) + y(y_1 + y_2) - z(z_1 + z_2)). \end{aligned}$$

The solutions to this which lie on $a_1 a_2$ are

$$m_1 \equiv [x_1 + x_2 : y_1 + y_2 : z_1 + z_2] \qquad \text{and} \qquad m_2 \equiv [x_1 - x_2 : y_1 - y_2 : z_1 - z_2].$$

These points are distinct since

$$\begin{aligned} (x_1 + x_2)(y_1 - y_2) - (x_1 - x_2)(y_1 + y_2) &= 2(x_2 y_1 - x_1 y_2) \\ (x_1 + x_2)(z_1 - z_2) - (x_1 - x_2)(z_1 + z_2) &= 2(x_2 z_1 - x_1 z_2) \\ (y_1 + y_2)(z_1 - z_2) - (y_1 - y_2)(z_1 + z_2) &= 2(y_2 z_1 - y_1 z_2) \end{aligned}$$

and by assumption $a_1$ and $a_2$ are distinct, and $2 \neq 0$, so at least one of these expressions is non-zero.

Also $m_1 \perp m_2$ since

$$\begin{aligned} &(x_1 + x_2)(x_1 - x_2) + (y_1 + y_2)(y_1 - y_2) - (z_1 + z_2)(z_1 - z_2) \\ &= (x_1^2 + y_1^2 - z_1^2) - (x_2^2 + y_2^2 - z_2^2) = 0. \quad \blacksquare \end{aligned}$$

A **midline** of the vertex $\overline{L_1 L_2}$ is a line $M$ which passes through $L_1 L_2$ and satisfies

$$S(L_1, M) = S(L_2, M).$$

**Theorem 55 (Midlines)** *The vertex $\overline{L_1 L_2}$ has a midline precisely when $C(L_1, L_2) = 1 - S(L_1, L_2)$ is a square. In this case if $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$, then we can renormalize so that*

$$l_1^2 + m_1^2 - n_1^2 = l_2^2 + m_2^2 - n_2^2$$

*and then there are exactly two midlines, namely*

$$M_1 \equiv [l_1 + l_2 : m_1 + m_2 : n_1 + n_2] \qquad \text{and} \qquad M_2 \equiv [l_1 - l_2 : m_1 - m_2 : n_1 - n_2].$$

*Furthermore $M_1 \perp M_2$.*

**Proof.** This is dual to the Midpoints theorem. $\blacksquare$

In classical hyperbolic geometry over the real numbers, there is always exactly one midpoint, while there are two midlines, usually called *angle bisectors*. So the symmetry between midpoints and midlines is missing. In a future paper, we will see that this symmetry gives rise to many rich aspects of triangle geometry in the universal hyperbolic setting. This symmetry is also shared by elliptic geometry.

## 3.14 Mid formulas

In classical trigonometry, both planar, elliptic and hyperbolic, there are many formulas in which midpoints, midlines and the associated half-distances and half-angles figure prominently. The analog of a relation such as $\beta = 2\alpha$ between angles $\alpha$ and $\beta$ in classical hyperbolic geometry is a relation such as

$$S = S_2(R) \equiv 4R(1-R) \tag{41}$$

between the corresponding spreads $R$ and $S$. The same kind of relation holds between quadrances $q$ and $r$ if $q = q(a_1, a_2)$ and $r = q(a_1, m) = q(a_2, m)$ where $m$ is a midpoint of $\overline{a_1 a_2}$, so this is also the analog between the relation $d = 2c$ between distances. Because there are two midpoints, or midlines, we should not be surprised that (41) is a quadratic equation.

Just as bisecting an angle generally involves moving to an extension field, so finding $R$, given $S$ in (41), requires solving the quadratic equation

$$4\left(R - \frac{1}{2}\right)^2 = 1 - S.$$

So such an $R$ exists precisely when $1 - S$ is a *square*, as we have also seen in the Midline theorem. If we assume the existence of midpoints or midlines, in other words that certain quadrances $q$ or certain spreads $S$ have the property that they are one minus a square, then a *number of the basic formulas of the subject have alternate formulations, sometimes simpler*. We call such relations *mid formulas*.

Recall the definition of the *Triple spread function* in (15):

$$S(a,b,c) = (a+b+c)^2 - 2\left(a^2+b^2+c^2\right) - 4abc.$$

The identity

$$S(t-a,b,c) - S(a,t-b,c) = 4c(b-a)(1-t)$$

suggests that something special happens when $t = 1$, and in fact

$$S(1-a,b,c) = S(a,1-b,c) = S(a,b,1-c) = S(1-a,1-b,1-c).$$

Also

$$S(1-a,1-b,1-c) = S(a,b,c) + (2a-1)(2b-1)(2c-1).$$

The next results similarly rely on pleasant identities.

**Theorem 56 (Triple quad mid)** *Suppose that*

$$q_1 \equiv 4p_1(1-p_1) \qquad q_2 \equiv 4p_2(1-p_2) \qquad \textit{and} \qquad q_3 \equiv 4p_3(1-p_3).$$

*Then*

$$S(q_1,q_2,q_3) = 0$$

*precisely when either*

$$S(p_1,p_2,p_3) = 0 \qquad \textit{or} \qquad S(1-p_1,1-p_2,1-p_3) = 0.$$

**Proof.** This follows from the identity

$$S(q_1,q_2,q_3) = -16S(p_1,p_2,p_3)\,S(1-p_1,1-p_2,1-p_3). \quad \blacksquare$$

**Theorem 57 (Pythagoras mid)** *Suppose that*

$$q_1 \equiv 4p_1(1-p_1) \qquad q_2 \equiv 4p_2(1-p_2) \qquad \textit{and} \qquad q_3 \equiv 4p_3(1-p_3).$$

*Then*

$$q_3 = q_1 + q_2 - q_1 q_2$$

*precisely when either*

$$p_3 = p_1 + p_2 - p_1 p_2 \qquad \textit{or} \qquad p_3 = 2p_1 p_2 - p_1 - p_2 + 1.$$

**Proof.** This follows from the identity

$$q_3 - q_1 - q_2 + q_1 q_2 = -4\left(p_3 - p_1 - p_2 + 2p_1 p_2\right)\left(p_3 - 2p_1 p_2 + p_1 + p_2 - 1\right). \quad \blacksquare$$

The next result shows that the existence of midpoints converts the Cross law from a quadratic equation to two linear ones. Recall the definition of the Cross function of (35):

$$C\left(\mathcal{A}, q_1, q_2, q_3\right) \equiv \left(\mathcal{A} - q_1 - q_2 - q_3 + 2\right)^2 - 4\left(1 - q_1\right)\left(1 - q_2\right)\left(1 - q_3\right).$$

**Theorem 58 (Cross mid)** *Suppose that*

$$q_1 \equiv 4p_1\left(1 - p_1\right) \qquad q_2 \equiv 4p_2\left(1 - p_2\right) \qquad \text{and} \qquad q_3 \equiv 4p_3\left(1 - p_3\right).$$

*Then for any number $\mathcal{A}$,*

$$C\left(\mathcal{A}, q_1, q_2, q_3\right) = 0$$

*precisely when either*

$$\mathcal{A} = 4S\left(p_1, p_2, p_3\right) \qquad \text{or} \qquad \mathcal{A} = 4S\left(1 - p_1, 1 - p_2, 1 - p_3\right).$$

**Proof.** This follows from the identity

$$\begin{aligned} &\left(\mathcal{A} - q_1 - q_2 - q_3 + 2\right)^2 - 4\left(1 - q_1\right)\left(1 - q_2\right)\left(1 - q_3\right) \\ &= \left(\mathcal{A} - 4S\left(p_1, p_2, p_3\right)\right)\left(\mathcal{A} - 4S\left(1 - p_1, 1 - p_2, 1 - p_3\right)\right). \quad \blacksquare \end{aligned}$$

## 3.15 Quadrance and spread of a couple

The **quadrance** $q\left(\overline{aL}\right)$ **of a non-null couple** $\overline{aL}$ is

$$q\left(\overline{aL}\right) = 1 - q\left(a, L^{\perp}\right).$$

In case the couple is non-dual, this is by the Complementary quadrances theorem equal to $q\left(a, b\right)$ where $b$ is the base point of $\overline{aL}$.

The **spread** $S\left(\overline{aL}\right)$ **of a couple** $\overline{aL}$ is

$$S\left(\overline{aL}\right) = 1 - S\left(L, a^{\perp}\right).$$

In case the couple is non-dual, this is by the Complementary spreads theorem equal to $S\left(L, R\right)$ where $R$ is the parallel line of $\overline{aL}$. The Quadrance/spread theorem shows that

$$q\left(\overline{aL}\right) = S\left(\overline{aL}\right).$$

**Theorem 59 (Couple quadrance spread)** *Suppose that $\overline{aL}$ is a non-null couple with $a \equiv [x : y : z]$ and $L \equiv (l : m : n)$. Then*

$$q\left(\overline{aL}\right) = S\left(\overline{aL}\right) = \frac{\left(lx + my - nz\right)^2}{\left(x^2 + y^2 - z^2\right)\left(l^2 + m^2 - n^2\right)}.$$

**Proof.** By definition $q\left(\overline{aL}\right)$ is the product between $a$ and $L^{\perp}$, and also $S\left(\overline{aL}\right)$ is the cross between $L$ and $a^{\perp}$. These are equal to the given expression. $\blacksquare$

## 3.16 Spread polynomials

The spread polynomials $S_n\left(x\right)$ are a remarkable family of polynomials that replace the Chebyshev polynomials $T_n\left(x\right)$ of the first kind in rational trigonometry, and their importance extends also to hyperbolic geometry. To motivate their introduction, we investigate the effect of combining three equal quadrances. Recall that the Equal quadrances spreads theorem states that if $S\left(a, b, c\right) = 0$ and $a = b$, then $c = 0$ or $c = S_2\left(a\right) \equiv 4a\left(1 - a\right)$.

**Theorem 60 (Three equal quadrances)** *Suppose that $a, b$ and $c$ are numbers which satisfy $S\left(a, b, c\right) = 0$ and $b = S_2\left(a\right)$. Then*

$$c = a \qquad \text{or} \qquad c = a\left(3 - 4a\right)^2.$$

**Proof.** The identity

$$S(a, 4a(1-a), c) = (a-c)\left(c - 9a + 24a^2 - 16a^3\right)$$

shows that either $c = a$ or

$$c = 9a - 24a^2 + 16a^3 = a(3-4a)^2. \ \blacksquare$$

If we continue in this way we generate the **spread polynomials** $S_n(x)$, which were defined in [17] recursively over a general field, by

$$\begin{aligned} S_0(x) &\equiv 0 \\ S_1(x) &\equiv x \\ S_n(x) &\equiv 2(1-2x) S_{n-1}(x) - S_{n-2}(x) + 2x. \end{aligned} \tag{42}$$

The next theorem is taken directly from [17].

**Theorem 61 (Recursive spreads)** *The spread polynomials $S_n(x)$ have the property that for any number $x$ and any $n = 1, 2, 3, \cdots$,*

$$S(x, S_{n-1}(x), S_n(x)) = 0.$$

**Proof.** Fix a number $x$ and use induction on $n$. For $n = 1$ the statement follows from the Equal quadrances spreads theorem. For a general $n \geq 1$,

$$S(x, S_{n-1}(x), S_n(x)) = 0$$

precisely when

$$(x + S_{n-1}(x) + S_n(x))^2 = 2\left(x^2 + S_{n-1}^2(x) + S_n^2(x)\right) + 4x S_{n-1}(x) S_n(x) \tag{43}$$

while

$$S(x, S_n(x), S_{n+1}(x)) = 0$$

precisely when

$$(x + S_n(x) + S_{n+1}(x))^2 = 2\left(x^2 + S_n^2(x) + S_{n+1}^2(x)\right) + 4x S_n(x) S_{n+1}(x). \tag{44}$$

Rearrange and factor the difference between equations (43) and (44) to get

$$(S_{n+1}(x) - S_{n-1}(x))(S_{n+1}(x) - 2(1-2x) S_n(x) + S_{n-1}(x) - 2x) = 0$$

Thus (44) follows from (43) if

$$S_{n+1}(x) = 2(1-2x) S_n(x) - S_{n-1}(x) + 2x.$$

Since this agrees with the recursive definition of the spread polynomials, the induction is complete. $\blacksquare$

The coefficients of the spread polynomials are integers, with the coefficient of $x^n$ in $S_n(x)$ a power of four. It follows that the degree of $S_n(x)$ is $n$ over any field not of characteristic two. Here are the first few spread polynomials.

$$\begin{aligned} S_0(x) &= 0 \\ S_1(x) &= x \\ S_2(x) &= 4x - 4x^2 = 4x(1-x) \\ S_3(x) &= 9x - 24x^2 + 16x^3 = x(3-4x)^2 \\ S_4(x) &= 16x - 80x^2 + 128x^3 - 64x^4 = 16x(1-x)(1-2x)^2 \\ S_5(x) &= 25x - 200x^2 + 560x^3 - 640x^4 + 256x^5 = x\left(5 - 20x + 16x^2\right)^2 \\ S_6(x) &= 4x(1-x)(3-4x)^2(1-4x)^2 \\ S_7(x) &= x\left(7 - 56x + 112x^2 - 64x^3\right)^2. \end{aligned}$$

More details about these new polynomials can be found in [6]. They play an important role in regular stars and polygons which we hope to discuss in a future paper. They also have remarkable number-theoretic properties, some of which are even more interesting than those of the Chebyshev polynomials $T_n(x.)$

# 4 Special triangles and trilaterals

In this section we study a triangle $\overline{a_1a_2a_3}$ and its dual trilateral $\overline{L_1L_2L_3}$, so that

$$L_1 = a_2a_3 \qquad L_2 = a_1a_3 \qquad L_3 = a_1a_2$$

and

$$a_1 = L_2L_3 \qquad a_2 = L_1L_3 \qquad a_3 = L_1L_2.$$

The standard conventions throughout are that the quadrances are

$$q_1 \equiv q(a_2, a_3) \qquad q_2 \equiv q(a_1, a_3) \qquad q_3 \equiv q(a_1, a_2)$$

and the spreads are

$$S_1 \equiv S(L_2, L_3) \qquad S_2 \equiv S(L_1, L_3) \qquad S_3 \equiv S(L_1, L_2).$$

If the triangle $\overline{a_1a_2a_3}$ is nil, or equivalently the trilateral $\overline{L_1L_2L_3}$ is null, then at least one of the points $a_1, a_2, a_3$ is null, so that at least two of the quadrances $q_1, q_2, q_3$ are undefined. If the triangle $\overline{a_1a_2a_3}$ is null, or equivalently the trilateral $\overline{L_1L_2L_3}$ is nil, then at least one of the lines $L_1, L_2, L_3$ is null, so that at least two of the spreads $S_1, S_2, S_3$ are undefined. So the existence of the three quadrances is equivalent to the triangle being non-nil, while the existence of the three spreads is equivalent to the trilateral being non-nil.

We begin by studying *right triangles*, in particular the phenomenon of parallax, and Napier's Rules. Then formulas for *isosceles* and *equilateral triangles* are derived, including the *Equilateral relation.* Then we establish the main *theorems of proportion*, such as Menelaus' theorem and Ceva's theorem and their duals.

## 4.1 Right triangles, parallax and Napier's rules

**Theorem 62 (Thales)** *Suppose that $\overline{a_1a_2a_3}$ is a non-null non-nil right triangle with $S_3 = 1$. Then*

$$S_1 = \frac{q_1}{q_3} \qquad \text{and} \qquad S_2 = \frac{q_2}{q_3}.$$

**Proof.** This follows directly from the Spread law. ■

**Theorem 63 (Thales' dual)** *Suppose that $\overline{L_1L_2L_3}$ is a non-null non-nil right trilateral with $q_3 = 1$. Then*

$$q_1 = \frac{S_1}{S_3} \qquad \text{and} \qquad q_2 = \frac{S_2}{S_3}.$$

**Proof.** This is dual to Thales' theorem. ■

**Theorem 64 (Right parallax)** *If a right triangle $\overline{a_1a_2a_3}$ has spreads $S_1 \equiv 0$, $S_2 \equiv S \neq 0$ and $S_3 \equiv 1$, then it will have only one defined quadrance, namely*

$$q_1 = \frac{S-1}{S}.$$

**Proof.** If $S_1 = 0$ then by the Zero spread theorem $a_1$ is a null point, so $q_2$ and $q_3$ are undefined. Since $S_2$ and $S_3$ are by assumption non-zero, $a_2$ and $a_3$ are non-null points. The Cross dual law

$$(S_2S_3q_1 - S_1 - S_2 - S_3 + 2)^2 = 4(1 - S_1)(1 - S_2)(1 - S_3)$$

applies, and becomes

$$(Sq_1 + 1 - S)^2 = 0.$$

Thus

$$q_1 = \frac{S-1}{S}. \quad ■$$

Reciprocally, we may restate the conclusion as

$$S = \frac{1}{1 - q_1}.$$

**Theorem 65 (Right parallax dual)** *If a right trilateral $\overline{A_1A_2A_3}$ has quadrances $q_1 \equiv 0$, $q_2 \equiv q \neq 0$ and $q_3 \equiv 1$, then it will have only one defined spread, namely*

$$S_1 = \frac{q-1}{q}.$$

**Proof.** This is dual to the Right parallax theorem. ∎

The following theorem is of considerable practical importance.

**Theorem 66 (Napier's rules)** *Suppose that a right triangle $\overline{a_1a_2a_3}$ has quadrances $q_1, q_2$ and $q_3$, and spreads $S_1, S_2$ and $S_3 \equiv 1$. Then any two of the quantities $S_1, S_2, q_1, q_2, q_3$ determine the other three, solely by the three **basic equations** from Thales' theorem and Pythagoras' theorem:*

$$S_1 = \frac{q_1}{q_3} \qquad S_2 = \frac{q_2}{q_3} \qquad \text{and} \qquad q_3 = q_1 + q_2 - q_1q_2.$$

**Proof.** Given two of the quadrances, determine the third via Pythagoras' theorem $q_3 = q_1 + q_2 - q_1q_2$, and then Thales' theorem gives the spreads.

Given two spreads $S_1$ and $S_2$, use Pythagoras' theorem and the relations $q_1 = S_1q_3$ and $q_2 = S_2q_3$ to obtain

$$1 = S_1 + S_2 - S_1S_2q_3.$$

Thus

$$\begin{aligned} q_3 &= \frac{S_1 + S_2 - 1}{S_1S_2} \\ q_1 &= S_1q_3 = \frac{S_1 + S_2 - 1}{S_2} \\ q_2 &= S_2q_3 = \frac{S_1 + S_2 - 1}{S_1}. \end{aligned}$$

Given a spread, say $S_1$, and one of the quadrances, then there are three possibilities. If the given quadrance is $q_3$, then $q_1 = S_1q_3$ and

$$\begin{aligned} q_2 &= \frac{q_3 - q_1}{1 - q_1} = \frac{q_3(1 - S_1)}{1 - S_1q_3} \\ S_2 &= \frac{q_2}{q_3} = \frac{1 - S_1}{1 - S_1q_3}. \end{aligned}$$

If the given quadrance is $q_1$, then

$$q_3 = \frac{q_1}{S_1}$$

and

$$\begin{aligned} q_2 &= \frac{q_3 - q_1}{1 - q_1} = \frac{q_1(1 - S_1)}{S_1(1 - q_1)} \\ S_2 &= \frac{q_2}{q_3} = \frac{1 - S_1}{1 - q_1}. \end{aligned}$$

If the given quadrance is $q_2$, then substitute $q_1 = S_1q_3$ into Pythagoras' theorem to get

$$q_3 = S_1q_3 + q_2 - S_1q_2q_3.$$

So

$$\begin{aligned} q_3 &= \frac{q_2}{1 - S_1(1 - q_2)} \\ q_1 &= \frac{S_1q_2}{1 - S_1(1 - q_2)} \\ S_2 &= \frac{q_2}{q_3} = 1 - S_1(1 - q_2). \quad ∎ \end{aligned}$$

The various equations derived in this proof are the analogs of Napier's rules, and are *fundamental for hyperbolic trigonometry.* It is perhaps best to remember that all follow from the basic equations by elementary algebraic manipulations.

**Theorem 67 (Napier's dual rules)** *Suppose that a right trilateral $\overline{A_1A_2A_3}$ has quadrances $q_1, q_2$ and $q_3 \equiv 1$, and spreads $S_1, S_2$ and $S_3$. Then any two of the quantities $q_1, q_2, S_1, S_2, S_3$ determine the other three, solely by the three **basic equations** from Thales' dual theorem and Pythagoras' dual theorem:*

$$q_1 = \frac{S_1}{S_3} \qquad q_2 = \frac{S_2}{S_3} \qquad \text{and} \qquad S_3 = S_1 + S_2 - S_1S_2.$$

**Proof.** This is dual to Napier's rules. ∎

**Example 16** *Let's investigate the existence of a triangle $\overline{a_1a_2a_3}$ with spreads $S_1 \equiv 1/4$, $S_2 \equiv 1/2$ and $S_3 \equiv 1$, which correspond respectively to angles of $\pi/6$, $\pi/4$ and $\pi/2$ over suitable extension fields of $\mathbb{Q}$. Since $\overline{a_1a_2a_3}$ is a right triangle, Napier's rules hold, so that*

$$q_3 = \frac{S_1 + S_2 - 1}{S_1S_2} = -2 \qquad q_1 = S_1q_3 = -\frac{1}{2} \qquad q_2 = S_2q_3 = -1.$$

*The quadrea is then $\mathcal{A}(\overline{a_1a_2a_3}) = 1/2$. To construct such a triangle let's assume that $a_3 \equiv [0:0:1]$ and that $a_1 \equiv [x:0:1]$ for some $x$ and $a_2 \equiv [0:y:1]$ for some $y$. Then we must have*

$$q_2 = \frac{x^2}{x^2-1} = -1$$

*so that $x = \pm 1/\sqrt{2}$ and*

$$q_1 = \frac{y^2}{y^2-1} = -\frac{1}{2}$$

*so that $y = \pm 1/\sqrt{3}$. Thus over $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$ we can construct such a triangle.* ⋄

**Example 17** *The same spreads as in the previous example can be achieved also over a finite field. Since $5^2 = 2$ and $7^2 = 3$ in $\mathbb{F}_{23}$, over this field there is a triangle $\overline{a_1a_2a_3}$ with spreads $S_1 \equiv 1/4 = 6$, $S_2 \equiv 1/2 = 12$ and $S_3 \equiv 1$: choose*

$$a_1 \equiv [14:0:1] \qquad a_2 \equiv [0:10:1] \qquad a_3 \equiv [0:0:1]. \ \diamond$$

**Example 18** *Suppose that $\overline{a_1a_2a_3}$ is a non-null right triangle with $S_3 \equiv 1$. Let $b$ denote the base point of the couple $a_3(a_1a_2)$. Define the quadrances*

$$r_1 \equiv q(a_1, b) \qquad r_2 \equiv q(a_2, b) \qquad \text{and} \qquad r_3 \equiv q(a_3, b).$$

*By the Complimentary spreads theorem $S(a_3b, a_3a_2) = 1 - S_2 = S_1$. Then by Thales' theorem applied to the triangles $\overline{a_1a_2a_3}, \overline{a_1ba_3}$ and $\overline{ba_2a_3}$,*

$$S_1 = \frac{q_1}{q_3} = \frac{r_3}{q_2} = \frac{r_2}{q_1}$$

*so that*

$$r_3 = \frac{q_1q_2}{q_3} \qquad r_2 = \frac{q_1^2}{q_3} \qquad \text{and} \qquad r_1 = \frac{q_2^2}{q_3}$$

*the last by symmetry.* ⋄

## 4.2 Isosceles triangles

The results of this section have obvious duals which we leave the reader to formulate.

A triangle $\overline{a_1a_2a_3}$ is **isosceles** precisely when at least two of its quadrances are equal or at least two of its spreads are equal. If all are defined, then the two conditions are equivalent.

**Theorem 68 (Pons Asinorum)** *Suppose that a triangle $\overline{a_1a_2a_3}$ has quadrances $q_1, q_2$ and $q_3$, and spreads $S_1, S_2$ and $S_3$. Then $q_1 = q_2$ precisely when $S_1 = S_2$.*

**Proof.** This follows from the Spread law

$$\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}. \ ∎$$

**Theorem 69 (Isosceles right)** *If $\overline{a_1a_2a_3}$ is a non-nil isosceles triangle with two right spreads $S_1 = S_2 \equiv 1$, then also $q_1 = q_2 = 1$, and furthermore $S_3 = q_3$.*

**Proof.** If $S_1$ and $S_2$ are defined then $\overline{a_1a_2a_3}$ is a non-null triangle, so by the Zero quadrances theorem all the quadrances are non-zero. Thales' law shows that

$$1 = \frac{q_2}{q_1}$$

so $q_1 = q_2$. Pythagoras' theorem then gives

$$q_1 = q_2 + q_3 - q_2q_3 = q_1 + q_3 - q_1q_3$$

from which $q_1 = q_2 = 1$. Then the Spread law shows that $S_3 = q_3$. ■

**Theorem 70 (Isosceles mid)** *Suppose that an isosceles triangle $\overline{a_1a_2a_3}$ has quadrances $q_1 = q_2 \equiv q$ and $q_3$, and corresponding spreads $S_1 = S_2 \equiv S$ and $S_3$, and that the couple $\overline{a_3(a_1a_2)}$ is non-dual, with base point $b$. If*

$$r_1 \equiv q(a_1, b) \qquad r_2 \equiv q(a_2, b) \qquad \textit{and} \qquad r_3 \equiv q(a_3, b)$$

*then*

$$r_3 = Sq$$

*and*

$$r_1 = r_2 = \frac{q(1-S)}{1-Sq}.$$

**Proof.** Since the couple $\overline{a_3(a_1a_2)}$ is non-dual by assumption, there is by the Base point theorem a unique point $b$ which lies on $a_1a_2$ and for which $ba_3$ is perpendicular to $a_1a_2$. By Pythagoras' theorem

$$q_2 = r_1 + r_3 - r_1r_3$$

while Thales' theorem shows that

$$S = \frac{r_3}{q}$$

so that

$$r_3 = Sq.$$

If $r_3 = Sq = 1$ then $q_2 = 1$, so that in $\overline{a_1a_3b}$ the Spread law gives $S_1 = 1$. But then by symmetry $S_2 = 1$, and so both $a_1$ and $a_2$ are base points of $\overline{a_3(a_1a_2)}$, which is impossible. So $r_3 \neq 1$ and

$$r_1 = \frac{q - r_3}{1 - r_3} = \frac{q(1-S)}{1-Sq}.$$

By symmetry $r_1 = r_2$. ■

**Theorem 71 (Isosceles triangle)** *Suppose that an isosceles triangle $\overline{a_1a_2a_3}$ has quadrances $q_1 = q_2 \equiv q$ and $q_3$, and corresponding spreads $S_1 = S_2 \equiv S$ and $S_3$, and that the couple $\overline{a_3(a_1a_2)}$ is non-dual. Then*

$$q_3 = \frac{4(1-S)q(1-q)}{(1-Sq)^2} \qquad \textit{and} \qquad S_3 = \frac{4S(1-S)(1-q)}{(1-Sq)^2}.$$

*Furthermore $1 - q_3$ is a square.*

**Proof.** Using the notation of the Isosceles triangle mid theorem,

$$q_3 = S_2(r_1) = 4r_1(1-r_1)$$

which is

$$q_3 = 4 \times \frac{q(1-S)}{1-Sq} \times \frac{(1-q)}{1-Sq} = \frac{4(1-S)q(1-q)}{(1-Sq)^2}.$$

Use the Spread law

$$\frac{S_3}{q_3} = \frac{S}{q}$$

to get

$$S_3 = \frac{4S\,(1-S)\,(1-q)}{(1-Sq)^2}.$$

Then

$$1 - q_3 = 1 - \frac{4q\,(1-q)\,(1-S)}{(1-Sq)^2} = \frac{(Sq-2q+1)^2}{(1-Sq)^2}$$

so that $1-q_3$ is indeed a square. Alternatively, since $b$ is a midpoint of $\overline{a_1a_2}$, the Midpoints theorem also shows that $1-q_3$ is a square. ■

**Example 19** *Suppose that* $a_1 \equiv [a:0:1]$, $a_2 \equiv [-a:0:1]$ *and* $a_3 \equiv [0:b:1]$, *so the triangle* $\overline{a_1a_2a_3}$ *has quadrances*

$$q_1 = q_2 \equiv q = -\frac{a^2+b^2-a^2b^2}{(1-a^2)(1-b^2)} \qquad \text{and} \qquad q_3 = -\frac{4a^2}{(a^2-1)^2}$$

*and spreads*

$$S_1 = S_2 \equiv S = \frac{b^2\left(1-a^2\right)}{a^2+b^2-a^2b^2} \qquad \text{and} \qquad S_3 = \frac{4a^2b^2\left(1-b^2\right)}{\left(a^2+b^2-a^2b^2\right)^2}.$$

*You may check that the relations in the Isosceles triangle theorem are satisfied.* ◇

**Theorem 72 (Isosceles parallax)** *If* $\overline{a_1a_2a_3}$ *is a non-null isosceles triangle with* $a_1$ *a null point,* $q_1 \equiv q$ *and* $S_2 = S_3 \equiv S$, *then*

$$q = \frac{4\,(S-1)}{S^2}.$$

**Proof.** The quadrance $q_1$ is non-zero since $\overline{a_1a_2a_3}$ is by assumption non-null, and so $L_1 \equiv a_2a_3$ is a non-null line. Since $a_1$ is a null point, the couple $\overline{a_1L_1}$ is non-dual, and so by the Altitude line and Base point theorems has an altitude line $N$ and a base point $b$. Apply the Right parallax theorem to both $\overline{a_1a_2b}$ and $\overline{a_1a_3b}$ to get

$$q\,(a_1,b) = q\,(a_2,p) = \frac{S-1}{S} \equiv r.$$

By the Equal quadrances theorem,

$$q = 4r\,(1-r) = 4\left(\frac{S-1}{S}\right)\left(\frac{1}{S}\right) = \frac{4\,(S-1)}{S^2}. \quad ■$$

## 4.3 Equilateral triangles

A triangle is **equilateral** precisely when either all its quadrances are equal or all its spreads are equal. In case these are all defined, Pons Asinorum implies that these two conditions are equivalent. The following formula appeared in the Euclidean spherical case as Exercise 24.1 in [17].

**Theorem 73 (Equilateral)** *Suppose that a triangle* $\overline{a_1a_2a_3}$ *is equilateral with common quadrance* $q_1 = q_2 = q_3 \equiv q$, *and with common spread* $S_1 = S_2 = S_3 \equiv S$. *Then*

$$(1-Sq)^2 = 4\,(1-S)\,(1-q)\,. \tag{45}$$

**Proof.** If $Sq = 1$ then any point of the triangle is the dual of the opposite line, so that all the quadrances are equal to 1, so both sides of the Equilateral relation are zero. Otherwise the result is a consequence of the Isosceles triangle theorem, which gives

$$q_3 = q = \frac{4q\,(1-q)\,(1-S)}{(1-Sq)^2}$$

so that

$$(1-Sq)^2 = 4\,(1-S)\,(1-q)\,. \quad ■$$

The **equilateral relation** (45) is symmetric in $S$ and $q$. Note that the point $[-3,-3]$ satisfies the relation, and that the $q$-intercept and $S$-intercept are both $3/4$.

**Theorem 74 (Equilateral mid)** *Suppose that*

$$S \equiv 4R(1-R) \qquad \text{and} \qquad q \equiv 4p(1-p).$$

*Then*

$$(1-Sq)^2 = 4(1-S)(1-q)$$

*precisely when either*

$$4Rp = 1 \qquad \text{or} \qquad 4R(1-p) = 1 \qquad \text{or} \qquad 4p(1-R) = 1 \qquad \text{or} \qquad 4(1-R)(1-p) = 1.$$

**Proof.** This follows from the identity

$$\begin{aligned}&(1-Sq)^2 - 4(1-S)(1-q)\\ &= (4Rp-1)(4R(1-p)-1)(4p(1-R)-1)(4(1-R)(1-p)-1). \quad \blacksquare\end{aligned}$$

## 4.4 Triangle proportions

The following results are direct analogs of planar theorems in [17], and the proofs are similar. The Triangle proportions theorem is self-dual, while Menelaus' theorem and Ceva's theorem have separate duals.

**Theorem 75 (Triangle proportions)** *Suppose that $\overline{a_1a_2a_3}$ is a triangle with quadrances $q_1$, $q_2$ and $q_3$, spreads $S_1, S_2$ and $S_3$, and that $d$ is a non-null point lying on the line $a_1a_2$, distinct from $a_1$ and $a_2$. Define the quadrances $r_1 \equiv q(a_1, d)$ and $r_2 \equiv q(a_2, d)$, and the spreads $R_1 \equiv S(a_3a_1, a_3d)$ and $R_2 \equiv S(a_3a_2, a_3d)$. Then*

$$\frac{R_1}{R_2} = \frac{S_1}{S_2}\frac{r_1}{r_2} = \frac{q_1}{q_2}\frac{r_1}{r_2}.$$

**Proof.** Define also $r_3 \equiv q(a_3, d)$. The assumptions imply that all of the quadrances and spread defined are non-zero. In $\overline{da_2a_3}$ use the Spread law to get

$$\frac{S_2}{r_3} = \frac{R_2}{r_2}.$$

In $\overline{da_1a_3}$ use the Spread law to get

$$\frac{S_1}{r_3} = \frac{R_1}{s_1}.$$

Thus

$$r_3 = \frac{S_2r_2}{R_2} = \frac{S_1r_1}{R_1}$$

and rearrange to obtain

$$\frac{R_1}{R_2} = \frac{S_1}{S_2}\frac{r_1}{r_2}.$$

Since

$$\frac{S_1}{S_2} = \frac{q_1}{q_2}$$

this can be rewritten as

$$\frac{R_1}{R_2} = \frac{q_1}{q_2}\frac{r_1}{r_2}. \quad \blacksquare$$

**Theorem 76 (Menelaus)** *Suppose that $\overline{a_1a_2a_3}$ is a non-null triangle, and that $L$ is a non-null line meeting $a_2a_3$, $a_1a_3$ and $a_1a_2$ at the non-null points $d_1$, $d_2$ and $d_3$ respectively. Define the quadrances*

$$\begin{aligned}r_1 &\equiv q(a_2, d_1) \qquad & t_1 &\equiv q(d_1, a_3)\\ r_2 &\equiv q(a_3, d_2) \qquad & t_2 &\equiv q(d_2, a_1)\\ r_3 &\equiv q(a_1, d_3) \qquad & t_3 &\equiv q(d_3, a_2).\end{aligned}$$

*Then*

$$r_1r_2r_3 = t_1t_2t_3.$$

**Proof.** If one of the points $d_1, d_2, d_3$ is a point of the triangle, then both sides of the required equation are zero and we are done. So suppose this is not the case. Define the spreads between $L$ and the non-null lines $a_2a_3$, $a_1a_3$ and $a_1a_2$ to be respectively $R_1$, $R_2$ and $R_3$. These are all non-zero since $d_1, d_2$ and $d_3$ are by assumption non-null, while the assumption on $\overline{a_1a_2a_3}$ ensures that all the quadrances involved in the theorem are also non-zero. So we can use the Spread law in the triangles $\overline{d_1d_2a_3}$, $\overline{d_2d_3a_1}$ and $\overline{d_3d_1a_2}$ to get

$$\frac{R_1}{R_2} = \frac{r_2}{t_1} \qquad \frac{R_2}{R_3} = \frac{r_3}{t_2} \qquad \frac{R_3}{R_1} = \frac{r_1}{t_3}.$$

Multiply these three equations to obtain

$$r_1r_2r_3 = t_1t_2t_3. \quad \blacksquare$$

The following result in the planar case was called the *Alternate spreads theorem* in [17].

**Theorem 77 (Menelaus' dual)** *Suppose that $\overline{A_1A_2A_3}$ is a non-null trilateral, and that $a$ is a non-null point joining $A_2A_3$, $A_1A_3$ and $A_1A_2$ to form the non-null lines $D_1$, $D_2$ and $D_3$ respectively. Define the spreads*

$$\begin{aligned} R_1 &\equiv S(A_2, D_1) \qquad & T_1 &\equiv S(D_1, A_3) \\ R_2 &\equiv S(A_3, D_2) \qquad & T_2 &\equiv S(D_2, A_1) \\ R_3 &\equiv S(A_1, D_3) \qquad & T_3 &\equiv S(D_3, A_2). \end{aligned}$$

*Then*

$$R_1R_2R_3 = T_1T_2T_3.$$

**Proof.** This is dual to Menelaus' theorem. $\blacksquare$

**Theorem 78 (Ceva)** *Suppose that $\overline{a_1a_2a_3}$ is a non-nil triangle, and that $a_0$ is a non-null point distinct from $a_1, a_2$ and $a_3$, and that the lines $a_0a_1$, $a_0a_2$ and $a_0a_3$ are non-null and meet the lines $a_2a_3$, $a_1a_3$ and $a_1a_2$ respectively at the points $d_1$, $d_2$ and $d_3$. Define the quadrances*

$$\begin{aligned} r_1 &\equiv q(a_2, d_1) \qquad & t_1 &\equiv q(d_1, a_3) \\ r_2 &\equiv q(a_3, d_2) \qquad & t_2 &\equiv q(d_2, a_1) \\ r_3 &\equiv q(a_1, d_3) \qquad & t_3 &\equiv q(d_3, a_2). \end{aligned}$$

*Then*

$$r_1r_2r_3 = t_1t_2t_3.$$

**Proof.** If one of the lines of $\overline{a_1a_2a_3}$ is null, then both sides of the required equation are zero. Otherwise we may assume that $\overline{a_1a_2a_3}$ is non-null, with $S_1, S_2$ and $S_3$ the usual spreads. Define the spreads

$$\begin{aligned} R_1 &\equiv S(a_1a_2, a_1a_0) \qquad & T_1 &\equiv S(a_1a_3, a_1a_0) \\ R_2 &\equiv S(a_2a_3, a_2a_0) \qquad & T_2 &\equiv S(a_2a_1, a_2a_0) \\ R_3 &\equiv S(a_3a_1, a_3a_0) \qquad & T_3 &\equiv S(a_3a_2, a_3a_0). \end{aligned}$$

as in the Alternate spreads theorem. Since $\overline{a_1a_2a_3}$ is non-nil, these are all non-zero. Then use the Triangle proportions theorem with the triangle $\overline{a_1a_2a_3}$ and the respective lines $a_1d_1$, $a_2d_2$ and $a_3d_3$ to obtain

$$\frac{R_1}{P_1} = \frac{S_2}{S_3}\frac{r_1}{t_1} \qquad \frac{R_2}{P_2} = \frac{S_3}{S_1}\frac{r_2}{t_2} \qquad \frac{R_3}{P_3} = \frac{S_1}{S_2}\frac{r_3}{t_3}.$$

Multiply these three equations and use the Alternate spreads theorem to get

$$\frac{R_1R_2R_3}{P_1P_2P_3} = \frac{r_1r_2r_3}{t_1t_2t_3} = 1. \quad \blacksquare$$

The dual of Ceva's theorem is not familiar in the Euclidean situation, where it holds also.

**Theorem 79 (Ceva's dual)** *Suppose that $\overline{A_1A_2A_3}$ is a non-nil trilateral and that $A_0$ is a non-null line distinct from $A_1, A_2$ and $A_3$, and that the points $A_0A_1$, $A_0A_2$ and $A_0A_3$ are non-null and join the points $A_2A_3$, $A_1A_3$ and $A_1A_2$ respectively to get the lines $D_1$, $D_2$ and $D_3$. Define the spreads*

$$\begin{aligned} R_1 &\equiv S(A_2, D_1) \qquad & T_1 &\equiv S(D_1, A_3) \\ R_2 &\equiv S(A_3, D_2) \qquad & T_2 &\equiv S(D_2, A_1) \\ R_3 &\equiv S(A_1, D_3) \qquad & T_3 &\equiv S(D_3, A_2). \end{aligned}$$

*Then*

$$R_1R_2R_3 = T_1T_2T_3.$$

**Proof.** This is dual to Ceva's theorem. ■

The converses of these theorems are not generally valid.

# 5 Null trigonometry

One of the significant differences between universal hyperbolic geometry and classical hyperbolic geometry is that the rich theory of *null trigonometry* plays a larger role. What is presented here is just an introduction to this fascinating subject, which has no parallel in Euclidean geometry. Formulations of dual results are left to the reader.

## 5.1 Singly nil triangles

Recall that a triangle is *singly nil* precisely when exactly one of its points is null.

**Theorem 80 (Nil cross law)** *Suppose the triangle $\overline{a_1a_2a_3}$ is singly nil, with $a_3$ a null point, spreads $S_1 \equiv S(a_1a_2, a_1a_3)$, $S_2 \equiv S(a_1a_2, a_2a_3)$ and $S_3 \equiv S(a_1a_3, a_2a_3)$, and quadrance $q_3 \equiv q(a_1, a_2)$. Then $S_3 = 0$ and*

$$q_3^2 - 2q_3\left(\frac{1}{S_1} + \frac{1}{S_2} - \frac{1}{S_1S_2}\right) + \left(\frac{1}{S_1} - \frac{1}{S_2}\right)^2 = 0.$$

*Furthermore $(1 - 2S_1)(1 - 2S_2)$ is a square.*

**Proof.** The Zero spread theorem shows that since $a_3$ is a null point, $S_3 = 0$, while since $a_1$ and $a_2$ are non-null points, $S_1$ and $S_2$ are non-zero. In this case the Cross dual law

$$(S_1S_2q_3 - (S_1 + S_2 + S_3) + 2)^2 = 4(1 - S_1)(1 - S_2)(1 - S_3)$$

still applies, and after rearrangement

$$S_1^2S_2^2q_3^2 - 2q_3S_2S_1(S_1 + S_2 - 2) + (S_2 - S_1)^2 = 0.$$

This can be rewritten as

$$q_3^2 - 2q_3\left(\frac{1}{S_1} + \frac{1}{S_2} - \frac{1}{S_1S_2}\right) + \left(\frac{1}{S_1} - \frac{1}{S_2}\right)^2 = 0.$$

After completing the square, it becomes

$$\left(q_3 - \left(\frac{1}{S_1} + \frac{1}{S_2} - \frac{1}{S_1S_2}\right)\right)^2 = \frac{(1 - 2S_1)(1 - 2S_2)}{S_1^2S_2^2}$$

so $(1 - 2S_1)(1 - 2S_2)$ must be a square. ■

**Example 20** *In the special case $S_2 \equiv 1$, the Nil cross law above becomes*

$$(S_1q_3 - S_1 + 1)^2 = 0$$

*so that*

$$q_3 = \frac{S_1 - 1}{S_1}$$

*as also given by the Right parallax theorem.* ◇

**Example 21** *In the special case $S_1 = S_2 \equiv S$, the Nil cross law becomes*

$$S^2q_3\left(S^2q_3 - 4S + 4\right) = 0$$

*so that $q_3 = 0$ or*

$$q_3 = \frac{4(S - 1)}{S^2}$$

*as also given by the Isosceles parallax theorem.* ◇

**Example 22** *Suppose that* $a_3 \equiv [1:0:0]$ *and that* $a_1 \equiv [x:0:1]$ *and* $a_2 \equiv [0:y:1]$*. Then*

$$q_3 = -\frac{\left(x^2+y^2-x^2y^2\right)}{\left(x^2-1\right)\left(y^2-1\right)}$$

*while*

$$S_1 = \frac{\left(1-x^2\right)y^2}{x^2+y^2-x^2y^2} \qquad \text{and} \qquad S_2 = \frac{\left(1-x\right)^2 y^2\left(1-y^2\right)}{x^2+y^2-x^2y^2}.$$

*You may check that these expressions satisfy the Nil cross law.* $\diamond$

## 5.2 Doubly nil triangles

**Theorem 81 (Doubly nil triangle)** *Suppose the triangle* $\overline{a_1a_2a_3}$ *is doubly nil, with* $a_1$ *and* $a_2$ *null points. Let* $h$ *be the quadrance of the couple* $\overline{a_3(a_1a_2)}$*. Then* $S_3 \equiv S(a_3a_1, a_3a_2)$ *and* $h$ *satisfy the relation*

$$S_3 = -\frac{4h}{\left(1-h\right)^2}.$$

**Proof.** The Parametrization of null points theorem shows that since $a_1$ and $a_2$ are null points, we can write

$$\begin{aligned} a_1 &= \alpha\left(t_1:u_1\right) \equiv \left[t_1^2-u_1^2 : 2t_1u_1 : t_1^2+u_1^2\right] \\ a_2 &= \alpha\left(t_2:u_2\right) \equiv \left[t_2^2-u_2^2 : 2t_2u_2 : t_2^2+u_2^2\right]. \end{aligned}$$

By the Join of null points theorem,

$$L_3 \equiv a_1a_2 = \left(t_1t_2-u_1u_2 : t_1u_2+t_2u_1 : t_1t_2+u_1u_2\right).$$

Now suppose that $a_3 = [x:y:z]$ for some numbers $x, y$ and $z$. Then from the Couple quadrance spread theorem and the identity (6)

$$h = \frac{\left(\left(t_1t_2-u_1u_2\right)x+\left(t_1u_2+t_2u_1\right)y-\left(t_1t_2+u_1u_2\right)z\right)^2}{\left(x^2+y^2-z^2\right)\left(t_1u_2-t_2u_1\right)^2}.$$

Also the spread $S_3$ is computed to be

$$S_3 = -\frac{4\left(x\left(t_1t_2-u_1u_2\right)+y\left(t_1u_2+t_2u_1\right)-z\left(t_1t_2+u_1u_2\right)\right)^2\left(t_1u_2-t_2u_1\right)^2\left(x^2+y^2-z^2\right)}{\left(x\left(t_1^2-u_1^2\right)+2yt_1u_1-z\left(t_1^2+u_1^2\right)\right)^2\left(x\left(t_2^2-u_2^2\right)+2yt_2u_2-z\left(t_2^2+u_2^2\right)\right)^2}.$$

It is then an algebraic identity that

$$S_3 = -\frac{4h}{\left(1-h\right)^2}. \quad \blacksquare$$

In a future paper we will see that this result has a natural interpretation in terms of spreads subtended by certain circles.

## 5.3 Triply nil triangles

**Theorem 82 (Triply nil quadreal)** *Suppose that* $\overline{\alpha_1\alpha_2\alpha_3}$ *is a triply nil triangle. Then*

$$\mathcal{L}\left(\overline{\alpha_1\alpha_2\alpha_3}\right) = -4.$$

**Proof.** Suppose that

$$\begin{aligned} \alpha_1 &= \alpha\left(t_1:u_1\right) \equiv \left[t_1^2-u_1^2 : 2t_1u_1 : t_1^2+u_1^2\right] \\ \alpha_2 &= \alpha\left(t_2:u_2\right) \equiv \left[t_2^2-u_2^2 : 2t_2u_2 : t_2^2+u_2^2\right] \\ \alpha_3 &= \alpha\left(t_3:u_3\right) \equiv \left[t_3^2-u_3^2 : 2t_3u_3 : t_3^2+u_3^2\right]. \end{aligned}$$

Then by the Join of null points theorem, the lines of $\overline{\alpha_1\alpha_2\alpha_3}$ are

$$\begin{aligned} L_1 &\equiv \alpha_2\alpha_3 = L\left(t_2 : u_2 | t_3 : u_3\right) \equiv \left(t_2t_3 - u_2u_3 : t_2u_3 + t_3u_2 : t_2t_3 + u_2u_3\right) \\ L_2 &\equiv \alpha_1\alpha_3 = L\left(t_1 : u_1 | t_3 : u_3\right) \equiv \left(t_1t_3 - u_1u_3 : t_1u_3 + t_3u_1 : t_1t_3 + u_1u_3\right) \\ L_3 &\equiv \alpha_1\alpha_2 = L\left(t_1 : u_1 | t_2 : u_2\right) \equiv \left(t_1t_2 - u_1u_2 : t_1u_2 + t_2u_1 : t_1t_2 + u_1u_2\right). \end{aligned}$$

A computer calculation shows that substituting these values into the expression (30) for $\mathcal{L}\left(L_1, L_2, L_3\right)$ gives identically the number $-4$. ■

**Theorem 83 (Triply nil balance)** *Suppose that $\overline{\alpha_1\alpha_2\alpha_3}$ is a triply nil triangle, and that $d$ is any point lying on $\alpha_1\alpha_2$ distinct from $\alpha_1$ and $\alpha_2$. If $b_1$ is the base of the couple $\overline{d\left(\alpha_1\alpha_3\right)}$, and $b_2$ is the base of the couple $\overline{d\left(\alpha_2\alpha_3\right)}$, then*

$$q\left(d, b_1\right) q\left(d, b_2\right) = 1.$$

*Furthermore $db_1$ is perpendicular to $db_2$.*

**Proof.** From the Parametrizing a line theorem we can find a proportion $r : s$ so that

$$d = \left[r\left(t_1^2 - u_1^2\right) + s\left(t_2^2 - u_2^2\right) : 2rt_1u_1 + 2st_2u_2 : r\left(t_1^2 + u_1^2\right) + s\left(t_2^2 + u_2^2\right)\right]. \tag{46}$$

Since $d$ is distinct from $\alpha_1$ and $\alpha_2$, both $r$ and $s$ are non-zero. The Join of null points theorem gives

$$\alpha_2\alpha_3 = \left(t_2t_3 - u_2u_3, t_2u_3 + t_3u_2, t_2t_3 + u_2u_3\right). \tag{47}$$

The quadrance $q\left(d, b_1\right)$ by the Couple quadrance spread theorem can be calculated from these two expressions, and with the aid of a computer we get

$$q\left(d, b_1\right) = -\frac{r\left(t_1u_3 - u_1t_3\right)^2}{s\left(t_2u_3 - t_3u_2\right)^2}.$$

Similarly

$$q\left(d, b_2\right) = -\frac{s\left(t_2u_3 - t_3u_2\right)^2}{r\left(t_1u_3 - u_1t_3\right)^2}.$$

So

$$q\left(d, b_1\right) q\left(d, b_2\right) = 1.$$

Applying the hyperbolic cross function $J$ to (46) and (47) we get an expression for the altitude line $db_1$ from $d$ to $\alpha_2\alpha_3$, and similarly an expression for $db_2$. Then applying the formula for the spread between two lines, a computer calculation shows that

$$S\left(db_1, db_2\right) = 1. \quad ■$$

As a consequence, Pythagoras' theorem gives

$$q\left(b_1, b_2\right) = q\left(d, b_1\right) + q\left(d, b_2\right) - 1.$$

The next theorem is but a brief introduction to a wealth of intricate relations that exist in a triply nil triangle. It suggests that there is a rich family of numbers that play a universal role in hyperbolic geometry, and raises the question of cataloguing such numbers.

**Theorem 84 (Triply nil orthocenter)** *Suppose that we work over a field with characteristic neither two, three or five, and that $\overline{\alpha_1\alpha_2\alpha_3}$ is a triply nil triangle, so that each of $\alpha_1, \alpha_2$ and $\alpha_3$ is a null point. Then the three sides of $\overline{\alpha_1\alpha_2\alpha_3}$ are non-null, and the three altitudes of the three couples of $\overline{\alpha_1\alpha_2\alpha_3}$ meet in a point $o$ called the* ***orthocenter*** *of $\overline{\alpha_1\alpha_2\alpha_3}$. If the bases of the altitudes are respectively $b_1, b_2$ and $b_3$, then $\overline{b_1b_2b_3}$ is an equilateral triangle with common quadrance $q = -5/4$ and common spread $S = 16/25$. The orthocenter of $\overline{b_1b_2b_3}$ is also $o$, and*

$$q\left(o, b_1\right) = q\left(o, b_2\right) = q\left(o, b_3\right) = -1/3.$$

**Proof.** Suppose that $\alpha_1 = \alpha(t_1 : u_1)$, $\alpha_2 = \alpha(t_2 : u_2)$ and $\alpha_3 = \alpha(t_3 : u_3)$. Then from the Join of null points theorem

$$\begin{aligned}\alpha_1\alpha_2 &= (t_1t_2 - u_1u_2 : t_1u_2 + t_2u_1 : t_1t_2 + u_1u_2)\\ \alpha_1\alpha_3 &= (t_1t_3 - u_1u_3 : t_1u_3 + u_1t_3 : t_1t_3 + u_1u_3)\\ \alpha_2\alpha_3 &= (t_2t_3 - u_2u_3 : t_2u_3 + u_2t_3 : t_2t_3 + u_2u_3).\end{aligned}$$

Use the Join of points and Meet of lines theorems to determine that the altitudes of $\overline{\alpha_1\alpha_2\alpha_3}$ meet at a point $o$, and to give explicit expressions for the points $b_1, b_2$ and $b_3$. While the exact formulas are somewhat lengthy to write down, the expressions for quadrance and spread applied to them, together with some pleasant computer simplifications, give the results. ■

Note that $q = -5/4$ and $S = 16/25$ satisfy the Equilateral relation $(1 - Sq)^2 = 4(1-S)(1-q)$.

## 5.4 Triangle thinness

One of the defining aspects of classical hyperbolic geometry is *thinness of triangles*. Here are two results that give universal approaches to this phenomenon.

**Theorem 85 (Triply nil Cevian thinness)** *Suppose that $\overline{\alpha_1\alpha_2\alpha_3}$ is a triply nil triangle, and that $a$ is a point distinct from $\alpha_1, \alpha_2$ and $\alpha_3$. Define the cevian points $c_1 \equiv (a\alpha_1)(\alpha_2\alpha_3)$, $c_2 \equiv (a\alpha_2)(\alpha_1\alpha_3)$ and $c_3 \equiv (a\alpha_3)(\alpha_1\alpha_2)$. Then*

$$\mathcal{A}(c_1, c_2, c_3) = 1.$$

**Proof.** Using the notation of the proofs of the previous theorems, suppose that $a \equiv [x : y : z]$ is an arbitrary point distinct from $\alpha_1, \alpha_2$ and $\alpha_3$. Then in terms of $t_1, u_1, t_2, u_2, t_3, u_3$ and $x, y$ and $z$, we may use the Joint of points and Meet of lines theorems to find expressions for $c_1, c_2$ and $c_3$, and then use a computer to evaluate the quadrea $\mathcal{A}(c_1, c_2, c_3)$. In terms of the expression (29) the numerator becomes the square of

$$\begin{aligned}&8(t_1u_2 - t_2u_1)(t_2u_3 - t_3u_2)(t_3u_1 - t_1u_3)(x(u_2u_3 - t_2t_3) - y(t_2u_3 + yt_3u_2) + z(t_2t_3 + u_2u_3))\\ &\times (x(t_1t_3 - u_1u_3) + y(t_1u_3 + u_1t_3) - z(t_1t_3 + u_1u_3))(x(t_1t_2 - u_1u_2) + y(t_1u_2 + t_2u_1) - z(t_1t_2 + u_1u_2))\end{aligned}$$

while the denominator has three factors which are all of the form

$$\begin{aligned}&4(t_3u_1 - t_1u_3)(t_1u_2 - t_2u_1)\\ &\times (x(t_1t_3 - u_1u_3) + y(t_1u_3 + u_1t_3) - z(t_1t_3 + u_1u_3))(x(t_1t_2 - u_1u_2) + y(t_1u_2 + t_2u_1) - z(t_1t_2 + u_1u_2)).\end{aligned}$$

The result then follows from pleasant cancellation. ■

**Theorem 86 (Triply nil altitude thinness)** *Suppose that $\overline{\alpha_1\alpha_2\alpha_3}$ is a triply nil triangle and that $a$ is a point distinct from the duals of the lines. If the altitudes to the lines of this triangle from $a$ meet the lines respectively at base points $b_1, b_2$ and $b_3$, then*

$$\mathcal{A}(b_1, b_2, b_3) = 1.$$

**Proof.** Using the notation of the proof of the previous theorem, the altitude line from $a$ to $\alpha_1\alpha_2$ is

$$N_3 \equiv [-y(t_1t_2 + u_1u_2) + z(t_1u_2 + t_2u_1) : x(t_1t_2 + u_1u_2) - z(t_1t_2 - u_1u_2) : x(t_1u_2 + t_2u_1) - y(t_1t_2 - u_1u_2)]$$

and its meet with $\alpha_1\alpha_2$ is the base point

$$b_3 = \begin{bmatrix} x\left(t_1^2 - u_1^2\right)\left(t_2^2 - u_2^2\right) + y(t_1u_2 + t_2u_1)(t_1t_2 - u_1u_2) - z\left(t_1^2t_2^2 - u_1^2u_2^2\right) \\ : x(t_1u_2 + t_2u_1)(t_1t_2 - u_1u_2) + 4yt_1t_2u_1u_2 - z(t_1t_2 + u_1u_2)(t_1u_2 + t_2u_1) \\ : x\left(t_1^2t_2^2 - u_1^2u_2^2\right) + y(t_1t_2 + u_1u_2)(t_1u_2 + t_2u_1) - z\left(t_1^2 + u_1^2\right)\left(t_2^2 + u_2^2\right) \end{bmatrix}$$

With similar expressions for $b_1$ and $b_2$, a computer calculation then shows that identically

$$\mathcal{A}(b_1, b_2, b_3) = 1. \quad ■$$

It is also worth noting that if the coordinates of $b_3$ in (**??**) are respectively $b_{31}, b_{32}$ and $b_{33}$, and similarly for $b_1$ and $b_2$, then

$$\det\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = -(t_1u_2 - t_2u_1)(t_2u_3 - t_3u_2)(t_3u_1 - t_1u_3).$$

## 5.5 Singly null singly nil triangles

A triangle $\overline{a_1a_2a_3}$ is **singly null and singly nil** when it has exactly one null point, and exactly one null line. There are two types of such triangles, depending on whether or not the null point lies on the null line.

**Theorem 87 (Singly null singly nil Thales)** *Suppose that $\overline{a_1a_2a_3}$ is a singly null and singly nil triangle in which $a_2a_3$ is a null line and $a_3$ is a null point. If $q_3 \equiv q(a_1, a_2)$ and $S_1 \equiv S(a_1a_2, a_1a_3)$ then*

$$q_3S_1 = 1.$$

**Proof.** Suppose that $a_3 = \alpha(t:u) \equiv \left[t^2-u^2 : 2tu : t^2+u^2\right]$ and $a_1 = [x:y:z]$, with $a_1$ non-null.

If $t^2+u^2 = 0$ then we can write $a_3 = [t:u:0]$, and by the Parametrizing a null line theorem $a_2 = [rt:ru:s]$ for some proportion $r:s$ where $s \neq 0$. In this case the definition of quadrance and the condition $t^2+u^2=0$ gives

$$q_3 = \frac{2r^2tuxy - 2rsuyz - 2rstxz + s^2x^2 + s^2y^2 - r^2t^2y^2 - r^2u^2x^2 + r^2t^2z^2 + r^2u^2z^2}{(x^2+y^2-z^2)\,s^2}$$

while the Spread formula gives

$$S_1 = \frac{\left(x^2+y^2-z^2\right)s^2}{2r^2tuxy - 2rsuyz - 2rstxz + s^2x^2 + s^2y^2 - r^2t^2y^2 - r^2u^2x^2 + r^2t^2z^2 + r^2u^2z^2}$$

so that

$$q_1S_1 = 1.$$

If $t^2+u^2 \neq 0$ then also by the Parametrizing a null line theorem we can write

$$a_2 \equiv \left(r\left(t^2-u^2\right) - 2stu : 2rtu + s\left(t^2-u^2\right) : r\left(t^2+u^2\right)\right)$$

for some proportion $r:s$ also with $s \neq 0$. Then a computer calculation shows also that

$$q_1S_1 = 1. \quad \blacksquare$$

This result suggests that if $a_2a_3$ is a null line and $a_3$ is a null point, then although $q(a_2, a_3)$ is not defined, it behaves in some respects like the number 1.

**Theorem 88 (Singly null singly nil orthocenter)** *Suppose that $\overline{a_1a_2a_3}$ is a triangle in which $a_1a_2$ is a null line, and that $a_3$ is a null point. Suppose that the base of the couple $\overline{a_1(a_2a_3)}$ is $b_1$, and the base of the couple $\overline{a_2(a_1a_3)}$ is $b_2$. Then the lines $a_1b_1$, $a_2b_2$ and $a_3(a_1a_2)^\perp$ intersect in a point $o$, and*

$$q(a_1, o) + q(o, b_1) = q(a_2, o) + q(o, b_2) = 1.$$

*Furthermore*

$$q(a_1, b_1) = -\left(2q(a_1, o) - 1\right)\left(2q(b_1, o) - 1\right).$$

**Proof.** This is a computer assisted calculation along the lines of the previous theorems. $\blacksquare$

Note that this gives a situation where the sum of two quadrances between three collinear points is 1, but no two of the points are necessarily perpendicular, as suggested by the proof of the Complementary quadrances spreads theorem.

## 5.6 Null perspective theorems

**Theorem 89 (Null perspective)** *Suppose that $\alpha_1, \alpha_2$ and $\alpha_3$ are distinct null points, and that $d$ is any point on $\alpha_1\alpha_3$ distinct from $\alpha_1$ and $\alpha_3$. Suppose further that $x$ and $y$ are points lying on $\alpha_1\alpha_2$ and that $z \equiv (\alpha_2\alpha_3)(xd)$ and $w \equiv (\alpha_2\alpha_3)(yd)$. Then*

$$q(x, y) = q(z, w).$$

**Proof.** This is a computer assisted calculation. $\blacksquare$

**Theorem 90 (Null subtended quadrance)** *Suppose that the line $L$ passes through the distinct null points $\alpha_1$ and $\alpha_2$. Then for a third null point $\alpha_3$, and a line $M$ distinct from $\alpha_1\alpha_3$ and $\alpha_2\alpha_3$, let $a_1 \equiv (\alpha_1\alpha_3) M$ and $a_2 \equiv (\alpha_2\alpha_3) M$. Then $q \equiv q(a_1, a_2)$ and $S \equiv S(L, M)$ are related by*

$$qS = 1.$$

*In particular $q$ is independent of $\alpha_3$.*

**Proof.** From the Parametrization of null points theorem we know that we can write $\alpha_1 = \alpha(t_1 : u_1)$, $\alpha_2 = \alpha(t_2 : u_2)$ and $\alpha_3 = \alpha(t_3 : u_3)$. Then from the Join of null points theorem

$$\begin{aligned} L \equiv \alpha_1\alpha_2 &= (t_1t_2 - u_1u_2 : t_1u_2 + t_2u_1 : t_1t_2 + u_1u_2) \\ \alpha_1\alpha_3 &= (t_1t_3 - u_1u_3 : t_1u_3 + t_3u_1 : t_1t_3 + u_1u_3) \\ \alpha_2\alpha_3 &= (t_2t_3 - u_2u_3 : t_2u_3 + t_3u_2 : t_2t_3 + u_2u_3). \end{aligned}$$

Suppose that $M = (l : m : n)$. Then computing with the Join of points theorem gives

$$\begin{aligned} a_1 &= (\alpha_1\alpha_3) M \\ &= [m(t_1t_3 + u_1u_3) - n(t_1u_3 + t_3u_1) : n(t_1t_3 - u_1u_3) - l(t_1t_3 + u_1u_3) : m(t_1t_3 - u_1u_3) - l(t_1u_3 + t_3u_1)] \end{aligned}$$

and

$$\begin{aligned} a_2 &= (\alpha_2\alpha_3) M \\ &= [m(t_2t_3 + u_2u_3) - n(t_2u_3 + t_3u_2) : n(t_2t_3 - u_2u_3) - l(t_2t_3 + u_2u_3) : m(t_2t_3 - u_2u_3) - l(t_2u_3 + t_3u_2)]. \end{aligned}$$

Then a computation using the definition of the quadrance between points gives

$$q \equiv q(a_1, a_2) = -\frac{(t_1u_2 - t_2u_1)^2 (l^2 + m^2 - n^2)}{(l(t_1^2 - u_1^2) + 2mt_1u_1 - n(t_1^2 + u_1^2))(l(t_2^2 - u_2^2) + 2mt_2u_2 - n(t_2^2 + u_2^2))}.$$

A computation using the definition of spread between lines gives

$$S \equiv S(L, M) = -\frac{\left(l\left(t_1^2 - u_1^2\right) + m2t_1u_1 - n\left(t_1^2 + u_1^2\right)\right)\left(l\left(t_2^2 - u_2^2\right) + 2mt_2u_2 - n\left(t_2^2 + u_2^2\right)\right)}{(t_1u_2 - t_2u_1)^2 (l^2 + m^2 - n^2)}.$$

Comparing these two expressions we see that

$$qS = 1. \quad \blacksquare$$

## 5.7 Four null points

Here are two theorems that will play an important role in the further development of the subject, and provide a link between hyperbolic geometry and the theory of cyclic quadrilaterals.

**Theorem 91 (Fully nil quadrangle diagonal)** *Suppose that $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$ are distinct null points, and that $d = (\alpha_1\alpha_2)(\alpha_3\alpha_4)$, $e = (\alpha_1\alpha_3)(\alpha_2\alpha_4)$ and $f = (\alpha_1\alpha_4)(\alpha_2\alpha_3)$. Then $d, e$ and $f$ are not collinear, and*

$$d^\perp = ef \qquad e^\perp = df \qquad \text{and} \qquad f^\perp = de.$$

**Proof.** From the Parametrization of null points theorem, we may write $\alpha_1 = \alpha(t_1 : u_1)$, $\alpha_2 = \alpha(t_2 : u_2)$, $\alpha_3 = \alpha(t_3 : u_3)$ and $\alpha_4 = \alpha(t_4 : u_4)$. The Null diagonal point theorem allows us to write down $d, e$ and $f$, namely

$$d = \left[\begin{array}{c} (t_1u_2 + t_2u_1)(t_3t_4 + u_3u_4) - (t_3u_4 + t_4u_3)(t_1t_2 + u_1u_2) \\ : (t_1t_2 + u_1u_2)(t_3t_4 - u_3u_4) - (t_3t_4 + u_3u_4)(t_1t_2 - u_1u_2) \\ : (t_1u_2 + t_2u_1)(t_3t_4 - u_3u_4) - (t_3u_4 + t_4u_3)(t_1t_2 - u_1u_2) \end{array}\right]$$

$$e = \left[\begin{array}{c} (t_1u_3 + u_1t_3)(t_2t_4 + u_2u_4) - (t_2u_4 + u_2t_4)(t_1t_3 + u_1u_3) \\ : (t_1t_3 + u_1u_3)(t_2t_4 - u_2u_4) - (t_2t_4 + u_2u_4)(t_1t_3 - u_1u_3) \\ : (t_1u_3 + u_1t_3)(t_2t_4 - u_2u_4) - (t_2u_4 + u_2t_4)(t_1t_3 - u_1u_3) \end{array}\right]$$

$$f=\left[\begin{array}{c}\left(t_1u_4+u_1t_4\right)\left(t_2t_3+u_2u_3\right)-\left(t_2u_3+u_2t_3\right)\left(t_1t_4+u_1u_4\right)\\ :\left(t_1t_4+u_1u_4\right)\left(t_2t_3-u_2u_3\right)-\left(t_2t_3+u_2u_3\right)\left(t_1t_4-u_1u_4\right)\\ :\left(t_1u_4+u_1t_4\right)\left(t_2t_3-u_2u_3\right)-\left(t_2u_3+u_2t_3\right)\left(t_1t_4-u_1u_4\right)\end{array}\right].$$

If the coefficients of $d$ above are $d_1, d_2$ and $d_3$ respectively, and similarly for $e$ and $f$, then a computer calculation shows that

$$\begin{aligned}&\det\begin{pmatrix}d_1 & d_2 & d_3\\ e_1 & e_2 & e_3\\ f_1 & f_2 & f_3\end{pmatrix}\\ &=-8\left(t_1u_2-t_2u_1\right)\left(t_2u_3-t_3u_2\right)\left(t_3u_4-t_4u_3\right)\left(t_4u_1-t_1u_4\right)\left(t_3u_1-t_1u_3\right)\left(t_4u_2-t_2u_4\right)\end{aligned}$$

so that $d, e$ and $f$ are non-collinear by the Collinear points theorem, since the four proportions are distinct.

Now $d$ and $e$ are perpendicular points because of the identity

$$\begin{aligned}&\left(\left(t_1u_2+t_2u_1\right)\left(t_3t_4+u_3u_4\right)-\left(t_3u_4+t_4u_3\right)\left(t_1t_2+u_1u_2\right)\right)\\ &\times\left(\left(t_1u_3+u_1t_3\right)\left(t_2t_4+u_2u_4\right)-\left(t_2u_4+u_2t_4\right)\left(t_1t_3+u_1u_3\right)\right)\\ &+\left(\left(t_1t_2+u_1u_2\right)\left(t_3t_4-u_3u_4\right)-\left(t_3t_4+u_3u_4\right)\left(t_1t_2-u_1u_2\right)\right)\\ &\times\left(\left(t_1t_3+u_1u_3\right)\left(t_2t_4-u_2u_4\right)-\left(t_2t_4+u_2u_4\right)\left(t_1t_3-u_1u_3\right)\right)\\ &-\left(\left(t_1u_2+t_2u_1\right)\left(t_3t_4-u_3u_4\right)-\left(t_3u_4+t_4u_3\right)\left(t_1t_2-u_1u_2\right)\right)\\ &\times\left(\left(t_1u_3+u_1t_3\right)\left(t_2t_4-u_2u_4\right)-\left(t_2u_4+u_2t_4\right)\left(t_1t_3-u_1u_3\right)\right)\\ &=0.\end{aligned}$$

Similarly $e$ and $f$ are perpendicular, and $d$ and $f$ are perpendicular. It follows that

$$d^{\perp}=ef \qquad e^{\perp}=df \qquad \text{and} \qquad f^{\perp}=de. \blacksquare$$

**Theorem 92** (48/64) *Suppose that $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$ are distinct null points, with diagonal spreads $P=S\left(\alpha_1\alpha_2, \alpha_3\alpha_4\right)$, $R=S\left(\alpha_1\alpha_3, \alpha_2\alpha_4\right)$ and $T=S\left(\alpha_1\alpha_4, \alpha_2\alpha_3\right)$. Then*

$$PR+RT+TP=48 \qquad \textit{and} \qquad PRT=64.$$

**Proof.** This is a computer calculation using the notation of the previous proof. $\blacksquare$

Note that as a consequence we have the relation

$$\frac{1}{P}+\frac{1}{R}+\frac{1}{T}=\frac{3}{4}.$$

Furthermore we find that the numbers 48 and 64 are constants of nature.

These results suggest that the theory of null quadrangles and quadrilaterals in universal hyperbolic geometry is quite rich, and this turns out to be the case, as will be discussed in a future paper.

# 6 Conclusion

Although it takes some getting used to, this new setting for such a venerable subject opens up many new directions for research, and encourages us to reconsider the *true role of algebraic geometry* in modern mathematics, where the term *algebraic geometry* is used in the rather broad sense of meaning an approach to geometry based on algebra.

Certainly one important direction is to use this understanding of hyperbolic geometry to begin a *deeper and more systematic exploration of relativistic geometries*, a subject that has seen remarkably little development in the hundred years since Einstein's introduction of special relativity, despite its obvious importance in understanding the world in which we live.

In subsequent papers I hope to also explore aspects of *triangle geometry, quadrilaterals, circles, isometries* and *tesselations* in the universal setting. Hopefully others will also find these topics attractive for investigation.

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