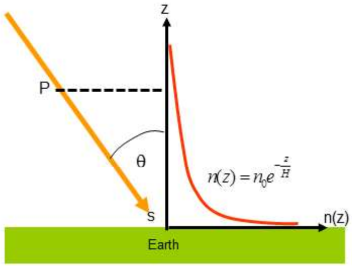
4.26 Radiative transfer - absorption only, exponential absorber

A 1-dimensional, plane parallel atmosphere is filled with a single absorbing gas whose vertical number density profile decays exponentially with height z according to $n(z) = n_0 \exp(-z/H)$ [molecules/cm^3], where the number density at the Earth's surface is n_0, the characteristic scale height is H and the height z increases from the surface at $z = 0$ to the top-of-the-atmosphere (TOA) at $z = \infty$. Assume that the monochromatic absorption cross section σ[cm^2] is independent of pressure and temperature and is therefore constant throughout the atmosphere.



a) Assume that our simplified atmosphere is illuminated from above by the sun under a solar zenith angle θ_S and we are interested in the shortwave spectral range. Calculate the downwelling radiance $I_\lambda^\downarrow(z)$ at a point P with height z. Neglect scattering and emission in the atmosphere, i.e. calculate how the incoming solar radiation is attenuated by *absorption* along the slant path from the sun to a certain height z in the atmosphere. Let the incoming solar radiance be $I_\lambda^\downarrow(\infty) \equiv I_0 = F_0\,\delta(\cos\theta - \cos\theta_S)\,\delta(\varphi - \varphi_S)$.

b) Write your result from above in the form $I_\lambda^\downarrow(z) = I_0\,T(z)$ and calculate the absorption rate i.e. $W(z) = |dT/dz|$. In case you could not solve part a) of this exercise, assume $T(z) = \exp(-\frac{\sigma n_0 H}{\cos\theta_S}\,e^{-\frac{z}{H}})$.

c) Calculate the height $z_{\max}$ where the absorption rate $W(z)$ is at its maximum. What is the corresponding optical path $\hat{\tau}_{\max}$ at this height?

d) How is the absorption rate $W(z)$ related to atmospheric heating by downwelling radiation?

4.26.1 Solution

a) In order to calculate the absorption of solar light in atmosphere of an exponentially with z decreasing absorber, we start with Lambert-Beer's law

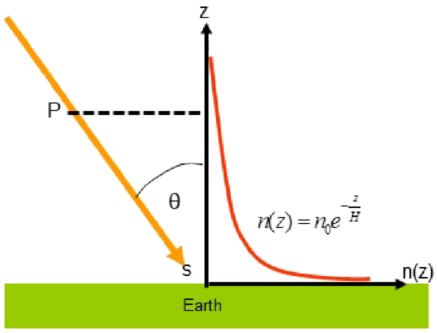


Figure 4.22: Graphical scheme of the absorption of solar radiation with an exponential aborber.

$$I^{\downarrow}(s) = I_o \cdot \exp\left(-\hat{\tau}(s)\right) \tag{4.186}$$

with the optical density $\hat{\tau}(s)$ being

$$\hat{\tau}(s) = \int_0^s \sigma_a(\lambda) \cdot n(s') \cdot ds' \tag{4.187}$$

with the absorption cross section $\sigma_a(\lambda)$ for a specific wavelength, the gas density $n(s') = n_o \cdot \exp(-z/H)$ (H being the scale height) and the path length $ds' = -dz/cos(\theta_S)$ (see the above figure). Inserting all definitions into equation 4.187 yields with an integration from z to infinity

$$\hat{\tau}(z) = \frac{\sigma_a(\lambda)}{cos(\theta_S)} \cdot \int_z^{\infty} n_o \cdot \exp(-z'/H) \cdot dz' = \frac{\sigma_a(\lambda) \cdot n_o \cdot H}{cos(\theta_S)} \cdot \exp(-z/H) \tag{4.188}$$

Inserting the solution 4.188 into equation 4.186 yields

$$\begin{aligned} I^{\downarrow}(z) &= I_o \cdot \exp\left(\frac{-\sigma_a(\lambda) \cdot n_o \cdot H}{cos(\theta_S)} \cdot \exp(-z/H)\right) && (4.189)\\ &= I_o \cdot T(z) && (4.190)\end{aligned}$$

with $T(z) := I^{\downarrow}(z)/I_o$

b) The absorption rate $W(z)$ is readily calculated from the derivative

$$W(z) = \frac{d(I^{\downarrow}(z)/I_o)}{dz} = \frac{dT(z)}{dz} = -\frac{\sigma_a(\lambda) \cdot n_o \cdot H}{cos(\theta_S)} \cdot \left(-\frac{1}{H}\right) \cdot \exp(-z/H) \cdot T(z) \tag{4.191}$$

hence

$$W(z) = \frac{\sigma_a(\lambda) \cdot n_o}{cos(\theta_S)} \cdot \exp(-z/H) \cdot T(z) \tag{4.192}$$

c) The absorption maximum is obtained from requiring $dW(z)/dz = 0$, i.e.,

$$\frac{dW(z)}{dz} = \frac{\sigma_a(\lambda) \cdot n_o}{cos(\theta_S)} \left(-\frac{1}{H} \cdot \exp(-z/H) \cdot T(z) + \exp(-z/H) \cdot \frac{dT(z)}{dz}\right) \tag{4.193}$$

and by inserting $\frac{dT(z)}{dz}$ from equation 4.191, we obtain

$$\frac{dW(z)}{dz} = \frac{\sigma_a(\lambda) \cdot n_o}{cos(\theta_S)} \cdot \exp(-z/H) \cdot T(z) \cdot \underbrace{\left(-\frac{1}{H} + \frac{\sigma_a(\lambda) \cdot n_o}{cos(\theta_S)} \cdot \exp(-z/H)\right)}_{0} \quad (4.194)$$

from which we calculate z_{max}

$$z_{max} = H \cdot \ln\left(\frac{\sigma_a(\lambda) \cdot n_o \cdot H}{cos(\theta_S)}\right) \quad (4.195)$$

and when inserting z_{max} into equation 4.188, $\hat{\tau}(z)$ is obtained

$$\hat{\tau}_{max}(z) = \frac{\sigma_a(\lambda) \cdot n_o \cdot H}{cos(\theta_S)} \cdot \exp(-z_{max}/H) = 1 \quad (4.196)$$

d) The gross solar heating rate H_{R} is given by:

$$H_{\mathrm{R}} = \frac{1}{\rho(z) C_p} \frac{\partial F^{\downarrow}(z)}{\partial z}, \quad (4.197)$$

with $F^{\downarrow}(z) = F_0 T(z)$ and

$$\frac{\partial F^{\downarrow}(z)}{\partial z} = F_0 \frac{dT}{dz} \quad (4.198)$$

$$= F_0 W(z). \quad (4.199)$$

using the irradiance $F^{\downarrow}(z)$ of the respective downwelling solar radiance $I^{\downarrow}_{\lambda}(z)$ and the air density $\rho(z)$ at level z according to the barometric height formula. C_p is the specific heat capacity of air at constant pressure (C_p = 1005 J/kg K). Thus, the gross solar heating rate H_{R} can be expressed as follows:

$$H_{\mathrm{R}} = \frac{F_0 W(z)}{\rho(z) C_p}. \quad (4.200)$$