A New Macroeconomic Modeling Platform Applied to Assess Effects of COVID-19*

Alexei Goumilevski⸸ Aquiles Farias⸸ Kadir Tanyeri⸸

June 18, 2021

Abstract

We have developed a flexible, powerful, and user-friendly platform for macroeconomic modeling in Python, including tools for filtering, simulation, estimation, forecasting and model diagnostics for Dynamic Stochastic General Equilibrium (DSGE) models. This platform can be applied for analysis of New Keynesian models, Real Business Cycle models, Gap models, and Overlapping Generations models, to name a few. It applies robust and efficient solution techniques to solve linear and nonlinear perfect foresight models which rely on rational expectations hypothesis. The system of non-linear model equations is solved with the aid of Michel Juillard et al. (1998) forward-backward substitution method. A novel feature of this software is an application of dynamic parameters to analyze models with structural changes which is crucial for policy analysis. For demonstration purposes we apply this Platform to study macroeconomic effects of COVID-19 pandemic on country economy. Our analysis utilizes Eichenbaum-Rebelo-Trabandt (2020) and Gali-Smets-Wouters (2012) models. ERT model is a non-linear model. It integrates the Neoclassical and the New Keynesian approaches with the theory of infection diseases. GSW model is a linear model. It incorporates unemployment theory developed by Gali (2011 a, b) into the new Keynesian model framework of Smets and Wouters (2007). The detrimental effects of epidemic on economy are modeled by an adverse shock to labor supply. We propose several scenarios of lockdown and vaccination policies and perform forecast simulations. These scenarios include policies that have different intensity and timing. These simulations produce scenarios forecasts, which can inform policy discussions in the context of surveillance and program review work.

JEL Classification Numbers: [E1,H0,I1]

Keywords: [DSGE, COVID -19, Vaccination, Lockdown]

---

*Prepared for the 27th International Conference on Computing in Economics and Finance

⸸International Monetary Fund

Contents

I. Introduction

In this paper we aim to estimate effects of COVID -19 epidemic on country economy. We start by analyzing some facts about this SARS disease rate of transmission and morbidity, then we apply Eichenbaum-Rebelo-Trabandt (2020 a) model to analyze possible effects of this virus on macroeconomic variables. ERT model lacks references to unemployment rate. We use Gali-Smets-Wouters (2012) model which embeds unemployment theory of Gali (2011 a, b). Effects of virus are accounted for by introducing a shock to labor supply.

Numerical calculations and subsequent analysis are accomplished with the aid of in-house developed Python software. This software is a quite versatile and flexible toolbox designed for DSGE modeling.

II. Literature Review

The literature on macroeconomic effects of this disease is constantly growing since the start of this pandemics at the beginning of 2020.

Mihailov (2020) studied macroeconomic effects of COVID-19 lockdown. Author considers scenarios of labor force reduction due to this virus for lockdown duration of one to three quarters. In the most optimistic case when ¼ of labor force is unable to work and lockdown duration is of one quarter, per capita loss in consumption is 6-7%, and per capita annual output loss is 3-4%. The recovery is of V shape and lasts 1.5-2 years. In the most pessimistic case of ¾ labor force reduction, loss in output could reach astonishing 12% and recovery could last for 10 years.

Cristina Arellano and Yan Bai (2020) integrated epidemiology concepts into sovereign debt default model. Authors notice, that lockdowns could alleviate health crisis but induce costly and prolonged debt crisis. Under the optimal policy output loss reaches 19%, lockdown lasts 8 months with intensity of 51%, and debt default crisis lasts for 43 months.

Eichenbaum, Rebelo, and Trabandt (2020 a, b) extended the canonical epidemiology model to study mutual effects of economic decisions and epidemic. While policy measure to reduce consumption and work hours alleviates health crisis counted by number of deaths, it exacerbates the size of the recession. Authors also incorporated concepts of treatment, vaccination, and containment into the DSGE framework and found optimal policy enter and exit times and containment duration.

Alvarez et al. (2020) studied optimal lockdown policy that controls number of infected and fatality cases while minimizing detrimental economic cost of lockdown. Authors investigated different lockdown regimes and observed that optimal policy could reduce number of infected at a peak by two times and number of deaths by 1% of population. Welfare under lockdown is lower and is equivalent to 2% of GDP one-time payment.

Caseli et al. (2020) applied panel regression methods to analyze data on effects of government lockdowns on mobility, social distancing, and COVID-19 infections. Authors conclude that lockdowns have significantly reduced economic activities that manifest itself thru its proxies, such as mobility and job postings. Authors also conclude that lockdowns are a powerful tool to reduce infections especially if applied at the early stages of epidemic and if they are sufficiently tight.

III. Python Platform

Macroeconomic models become more and more complex nowadays and, as a rule, are not tractable analytically. Several macroeconomic modeling frameworks have been developed to handle these models. Among these are: IRIS Macroeconomic Modeling Toolbox, Dynare software platform, and Troll software, to name a few. In this document we present a brief overview of Python software. This software could help economists and alike calibrate models, run simulations, and perform forecasts.

Some of the highlights are listed below:

- Platform is written in Python language and uses only Python libraries that are available by installing Anaconda distribution.

- Platform is versatile to parse model files written in a human readable YAML format, Sirius XML format and to parse simple IRIS and DYNARE model files.

- Prototype model files are created for non-linear and linear perfect-foresight models.

- Platform parses a model file and checks its syntax for errors. It generates Python functions source code and computes Jacobian up to the third order in a symbolic form.

- Non-linear equations are solved by iterations by Newton's method. Two algorithms are implemented: Armstrong et al. (1998) stacked matrices method and Juillard et al. (1998) forward-backward substitution method.

- Linear models are solved with Binder and Pesaran's method, Anderson and Moore's method, and two methods that reproduce calculations employed in Dynare and Iris software and use generalized Schur matrix decomposition algorithms.

- Platform can run forecasts with user's judgmental adjustments on a path of some or all endogenous variables.

- Non-linear models can be run with time dependent parameters.

- Platform can be used to calibrate models to find model's parameters of linear and nonlinear models. Platform applies Bayesian approach to maximize likelihood function that incorporates prior beliefs about parameters and goodness of fit of model to the data.

- Platform can sample model parameters by using Markov Chain Monte Carlo affine invariant ensemble sampler algorithm of Jonathan Goodman and adaptive Metropolis-Hastings algorithms of Paul Miles. The former algorithm is useful for sampling badly scaled distributions of parameters. The later algorithm employs adaptive Metropolis methods that incorporate delayed rejection to stimulate samples' states mixing.

- Platform uses Scientific Python Sparse package for large matrices algebra.

- Following filters are implemented: Kalman (linear and non-linear models), LRX, HP, Bandpass. Versions of Kalman filter and smoother algorithms include diffuse and non-diffuse, multi-variate and univariate filters.

IV. Epidemic Stylized Facts

COVID-19 disease is caused by Severe Acute Respiratory Syndrome coronavirus. It is an airborne virus that can spread through small droplets of saliva or indirectly via surfaces that have been touched by someone who was infected with this virus. Since its outbreak in March 2020, more than 166 million people worldwide had been infected and about 3.4 million had died. In the US 33 million people were infected, and 588 thousand died.

Below we present data on epidemic in the US and in four major EU economies. We sourced these data from the Center for System Science and Engineering at John Hopkins University[1]. Data on active COVID-19 cases in the United States were sourced from Worldometer's website[2].

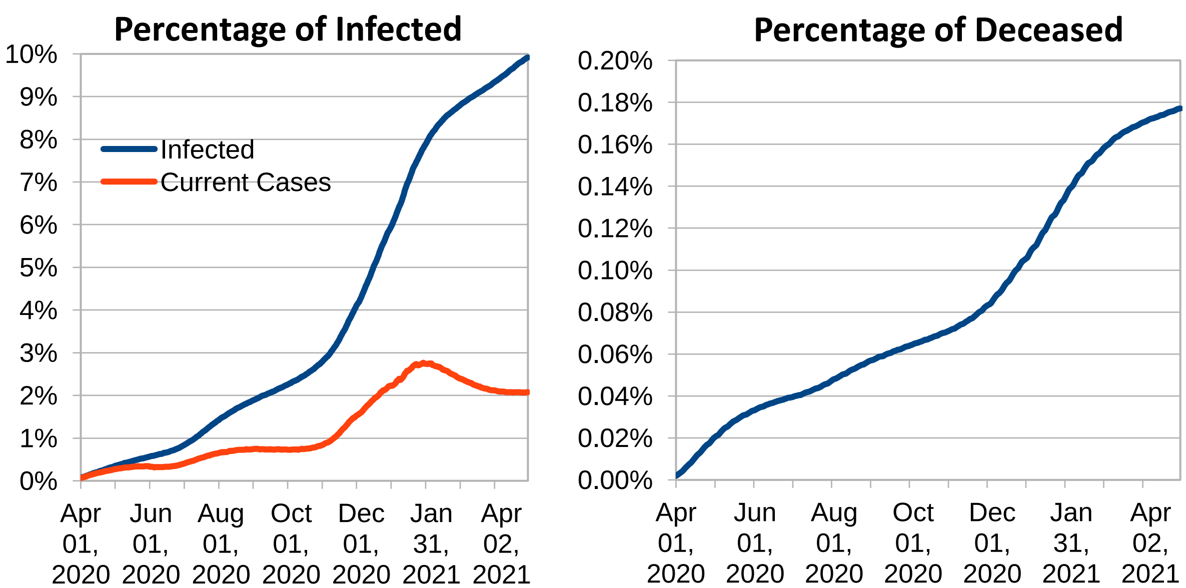

Fig.4.1. Dynamics of coronavirus epidemic in the US. Number of infected is the total number of reported cases of COVID-19 infection.

[1] Coronavirus Source Data, https://ourworldindata.org/coronavirus-source-data
[2] COVID-19 Cases in the United States, https://www.worldometers.info/coronavirus/country/us/#graph-cases-daily

Based on these data for US, Germany, France, Italy and Spain, we calculated daily rate of infection transmission as a ratio of daily change of the number of infected to the total number, and daily rate of deaths as a ratio of the number of daily deaths to the total number of infected. At the beginning of epidemic number of infected is small. Because these calculations assume division by the total number of infected and at the onset of epidemic it is small, we report daily rates starting from June 2020.

These data demonstrate a seasonal nature of COVID-19 virus: infection transmission increases in winter and it decreases in summer.

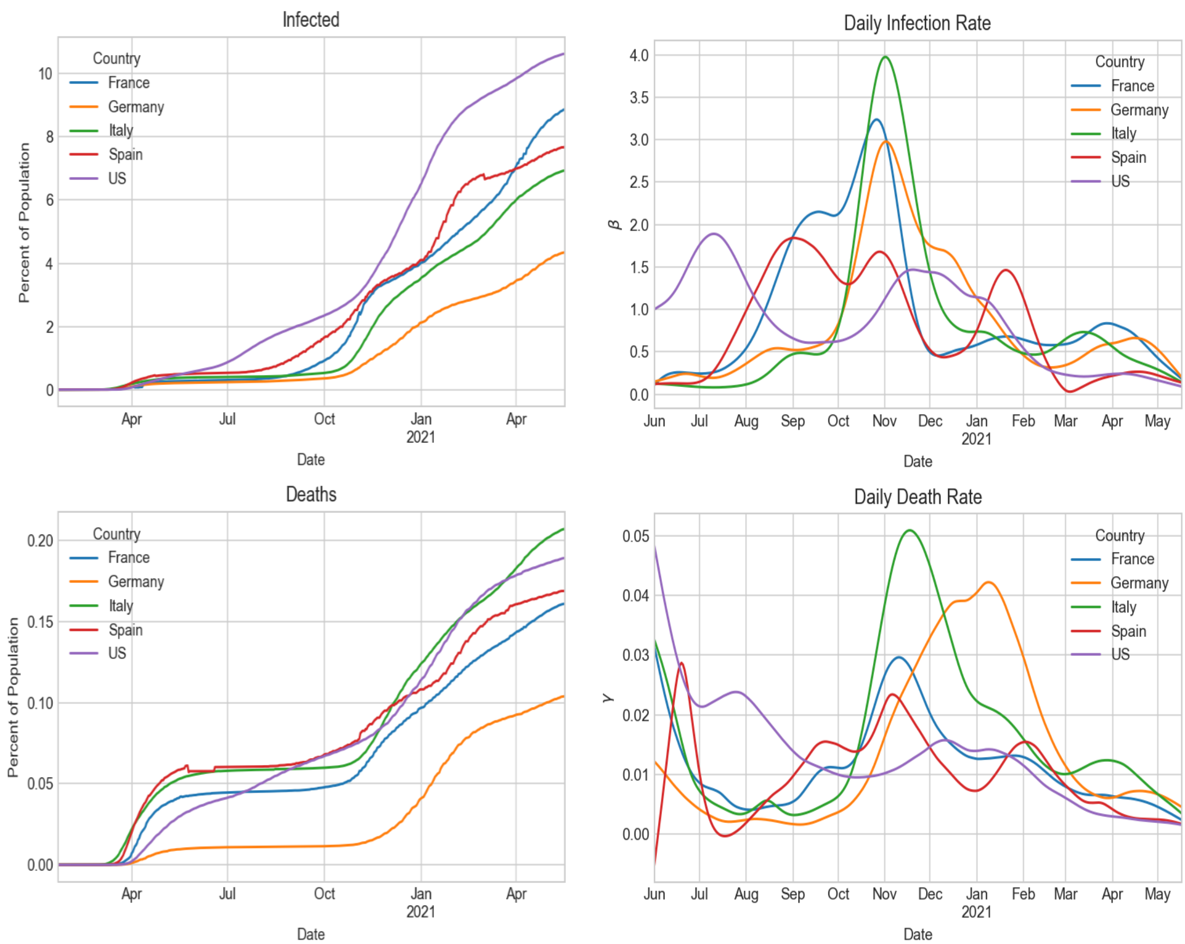

Fig.4.2. Percent of total cases of infected and deceased population in five major world economies. Daily rates of infected and deceased were filtered by applying HP filter to smooth the scattered data. Vertical axis displays percentage point.

In figure 4.3 we present major macroeconomic indicators[1] of US economy for period from January 2006 to March 2021. This time range includes Global Financial Crisis of 2007 and 2008 and economic recession of 2020. We computed output and consumption gaps as percentage deviations

[1] Source: World Economic Outlook and Haver Analytics databases.

of time series from HP filtered values. These data show a drop in output and consumption since epidemic outbreak followed by a V shape recovery. Similarly, unemployment rate, wage rate, average working hours per week, and total labor force show a quick recovery from COVID-19 disease impact.

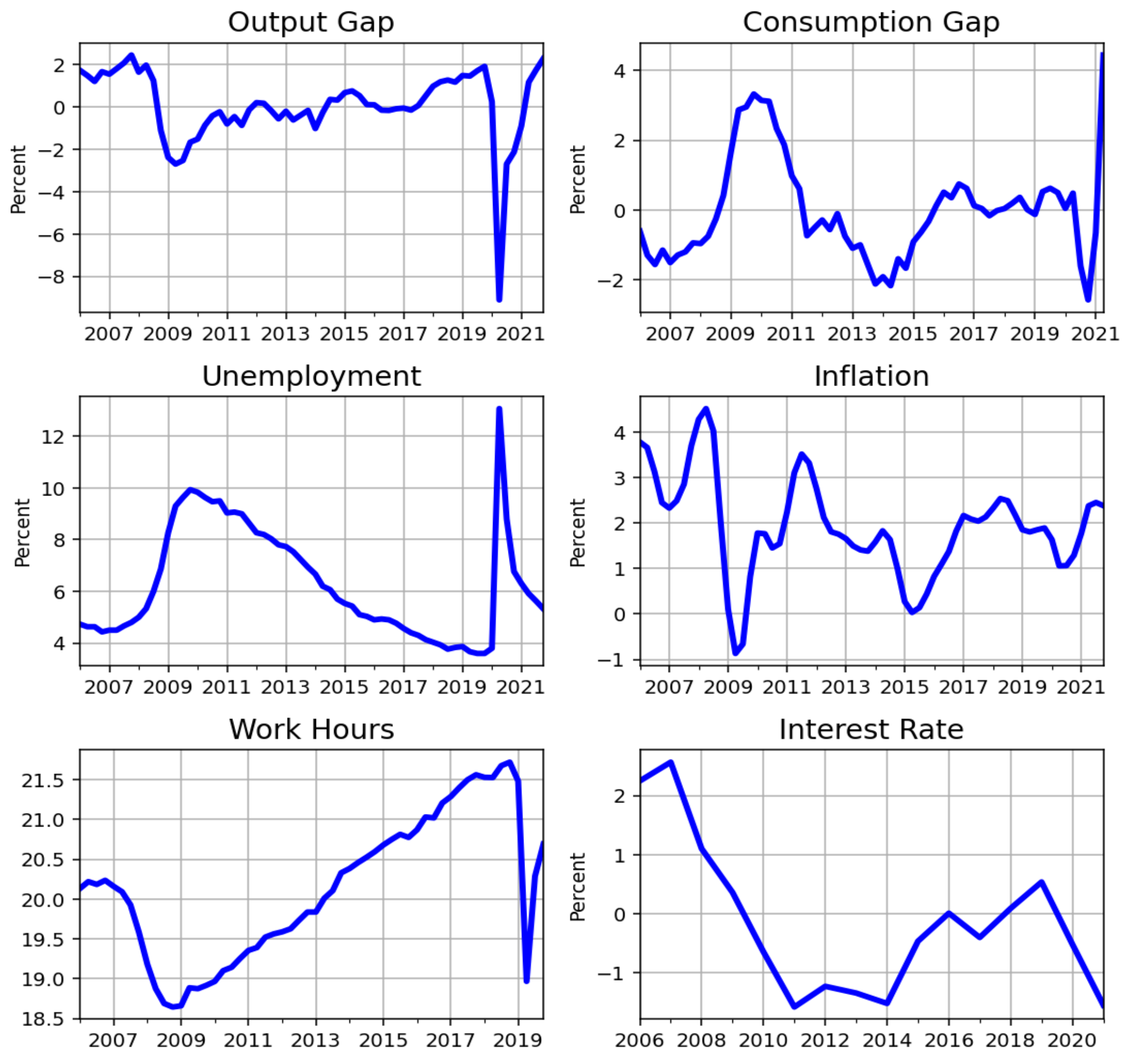

Fig.4.3. Macroeconomic indicators of US economy. Output gap, consumption gap, unemployment and inflation rates are shown in percentage points. Work hours are average hours per week. Interest rate is a six-month London interbank offered rate (LIBOR), period averaged.

V. Epidemiological Models

Epidemiological models describe spread of an epidemic. These compartmental models assume that an individual could transition thru several states during infection period: Susceptible, Infected, Recovered and Deceased. If incubation period of being exposed to infection, but still not infected, is large, then one could add Exposed state. These models also include models describing transmission of virus when individuals are vaccinated, and models with individuals' age structuring.

We consider a simple SIR model consisting of four equations,

$$\begin{aligned} \frac{dS}{dt} &= -\beta IS \\ \frac{dI}{dt} &= \beta IS - (\mu + \gamma)I \\ \frac{dR}{dt} &= \mu I \\ \frac{dD}{dt} &= \gamma I \end{aligned} \tag{5.1}$$

Here S is the stock of susceptible, I is the flow of infected, R is the stock of recovered and D is the stock of deceased population.

We assume that at time zero number of infected population is 0.05%. Rewriting equation for infected individuals as,

$$\frac{dI}{dt} = (R_b S - 1)(\mu + \gamma)I$$

it yields that if $R_b S(0) < 1$, then number of infected will decrease with time. Here the basic reproduction number is, $R_b = {}^{\beta}/_{\mu + \gamma}$. If $R_b \leq 1$, the disease can never cause a proper epidemic outbreak.

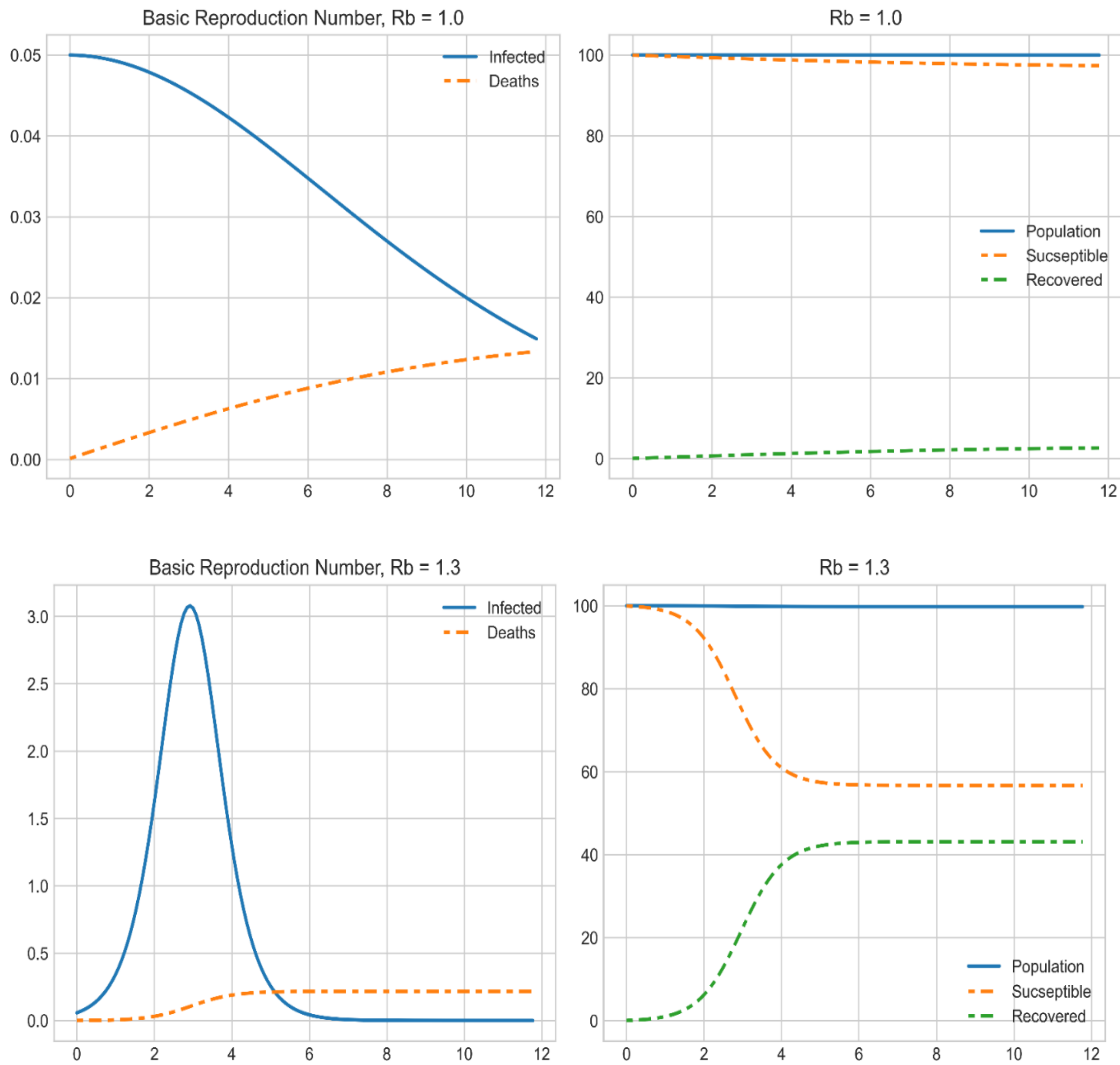

Fig.5.1. Diagram of the SIR model forecast. Initial infected population was, 0.05%. and the basic reproduction number R_b was, 1 and, 1.3. The Y axis shows percent of population, and the X axis displays time in quarters. For $R_b = 1$ the epidemic never occurs.

VI. ERT MODEL

In this section we describe Eichenbaum-Rebelo-Trabandt model. This model embeds epidemiological concepts into DSGE modelling framework. We start with SIR model equations. In addition to the standard channel of infection transmission, the number of newly infected population is affected by an economic activity of agents,

$$T_t = \pi_1 (S_t c_t^S)(I_t c_t^I) + \pi_2 (S_t n_t^S)(I_t n_t^I) + \pi_3 (S_t I_t) \quad (6.1)$$

The first two terms describe infection transmitted thru consumption and work channels.

Populations of susceptible, infected, recovered, and deceased evolve according to equations of the standard epidemiological model:

$$
\begin{aligned}
S_{t+1} &= S_t - T_t \\
I_{t+1} &= I_t + T_t - (\pi_R + \pi_D) I_t \\
R_{t+1} &= R_t + \pi_R I_t \\
D_{t+1} &= D_t + \pi_D I_t
\end{aligned} \tag{6.2}
$$

Model introduces macro variables of suspected, recovered, and infected individuals' such as consumption and work hours. The Cobb-Douglass type equation for output y_t, equations for aggregated consumption c_t, investment x_t, government spending g, marginal cost mc_t, and capital k_t are standard ones:

$$
\begin{aligned}
y_t &= p_t\, A\, k_t^{1-\alpha} n_t^{\alpha} \\
y_t &= c_t + x_t + g \\
mc_t &= A^{-1} \alpha^{-\alpha} (1-\alpha)^{\alpha-1} w_t^{\alpha} \left(r_t^k\right)^{1-\alpha} \\
w_t &= \alpha A mc_t n_t^{\alpha-1} k_t^{1-\alpha} \\
k_{t+1} &= x_t + (1-\delta) k_t
\end{aligned} \tag{6.3}
$$

Aggregate equations for consumption c_t and working hours n_t of suspected, infected, and recovered are,

$$
\begin{aligned}
c_t &= S_t c_t^S + I_t c_t^I + R_t c_t^R \\
n_t &= S_t n_t^S + I_t n_t^I + R_t n_t^R
\end{aligned} \tag{6.4}
$$

Authors introduce utility function of households,

$$
U = \sum_{t=0}^{\infty} \beta^t \left\{ S_t \left[\log(c_t^S) - \frac{\theta}{2} (n_t^S)^2 \right] + I_t \left[\log(c_t^I) - \frac{\theta}{2} (n_t^I)^2 \right] + R_t \left[\log(c_t^R) - \frac{\theta}{2} (n_t^R)^2 \right] \right\} \tag{6.5}
$$

It is subject to budget constraint of household family members,

$$
c_t + \psi = w_t n_t + r_t^k\, k_t + \varphi_t \tag{6.6}
$$

Here ψ is the lump-sum taxes, and φ_t is the firms' profit. The government finances spending with its income from taxes, i.e., $g = \psi$.

Eichenbaum et al. (2020a) derived first order conditions by equating derivatives of a Lagrange function to zero. Below we present equations for macroeconomic variables and for Lagrange multipliers λ:

$$
\begin{aligned}
1/c_t^S &= \lambda_t^b - \pi_1\, \lambda_t^{\tau}\, I_t\, c_t^I \\
1/c_t^I &= \lambda_t^b
\end{aligned}
$$

$$
\begin{aligned}
&{}^{1}/_{c_t^R} = \lambda_t^b \\
&\theta n_t^S = \lambda_t^b\, w_t + \pi_2\, \lambda_t^\tau\, I_t\, n_t^I \\
&\theta n_t^I = \lambda_t^b\, w_t \\
&\theta n_t^R = \lambda_t^b\, w_t \\
&\lambda_t^b = \beta\,\left(r_{t+1}^k + 1 - \delta\right)\lambda_{t+1}^b \\
&\lambda_t^I = \lambda_t^\tau + \lambda_t^S \\
&log(c_{t+1}^S) = \frac{\theta}{2}\,(n_{t+1}^S)^2 - \lambda_{t+1}^\tau[\pi_1 c_{t+1}^S(I_t c_t^I) + \pi_2 n_{t+1}^S(I_t n_t^I) + \pi_3 I_{t+1}\,] \\
&\qquad\qquad -\lambda_{t+1}^b[w_{t+1}n_{t+1}^S - c_{t+1}^S] + \frac{\lambda_t^S}{\beta} - \lambda_{t+1}^S \\
&log(c_{t+1}^I) = \frac{\theta}{2}\,(n_{t+1}^I)^2 - \lambda_{t+1}^b[w_{t+1}n_{t+1}^I - c_{t+1}^I] + \frac{\lambda_t^I}{\beta} - \lambda_{t+1}^I[1 - \pi_R - \pi_D] - \lambda_{t+1}^R\,\pi_R \\
&log(c_{t+1}^R) = \frac{\theta}{2}\,(n_{t+1}^R)^2 - \lambda_{t+1}^b[w_{t+1}n_{t+1}^R - c_{t+1}^R] + \frac{\lambda_t^R}{\beta} - \lambda_{t+1}^R \\
&\lambda_t^b = \beta\, r_t \lambda_{t+1}^b \\
&r_t = \frac{R_t^b}{\pi_{t+1}}
\end{aligned}
\tag{6.7}
$$

The optimality conditions for price settings are,

$$
\begin{aligned}
&K_t^f = \gamma\, mc_t\, \lambda_t^b\, y_t + \beta\xi\pi_{t+1}^{\hat{\gamma}/\hat{\gamma}-1} K_{t+1}^f \\
&F_t = \lambda_t^b\, y_t + \beta\xi\pi_{t+1}^{1/\hat{\gamma}-1} F_{t+1} \\
&K_t^f = F_t\left(\frac{1 - \xi\pi_t^{1/\hat{\gamma}-1}}{1-\xi}\right)^{1-\hat{\gamma}}
\end{aligned}
\tag{6.8}
$$

The price dispersion term is,

$$
p_t = \left[(1-\xi)\left(\frac{1-\xi\pi_t^{1/\hat{\gamma}-1}}{1-\xi}\right)^{\hat{\gamma}} + \xi\,\frac{\pi_t^{\hat{\gamma}/\hat{\gamma}-1}}{p_{t-1}}\right]^{-1} \tag{6.9}
$$

The Taylor rule equation reads,

$$
R_t^b = r_{ss} + \theta_\pi\, log\left(\frac{\pi_t}{\pi_{ss}}\right) + \theta_x\, log\left(\frac{y_t}{y_t^f}\right) \tag{6.10}
$$

Here θ_x is the Taylor rule coefficient for output gap, and y_t^f is the flexible price output which can be computed by setting $\xi = 0$.

A. Calibration

We calibrated SIR model parameters based on US COVID-19 data. These data display seasonal nature of rate of infection and deaths – these rates increase in winter and subside in spring. The US average weekly transmission and weekly death rates since May 2020 are: $\beta = 16\%, \gamma = 0.3\%$.

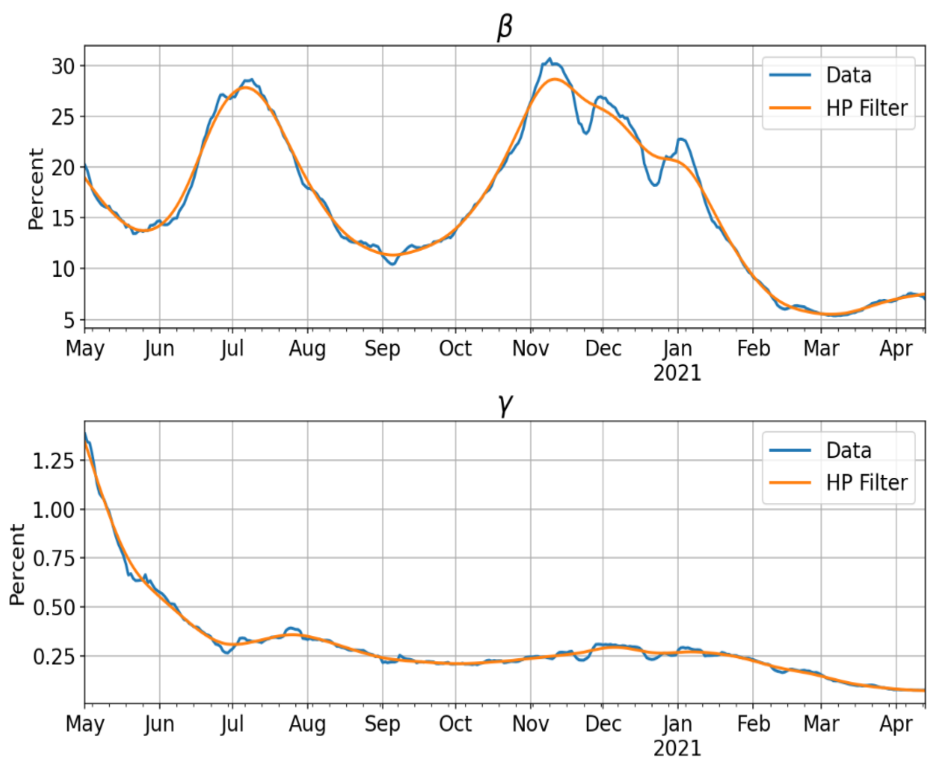

Fig.6.1. Rates of infection transmission and death in the United States. These rates are ratios of weekly changes of the number of current infection cases and death cases to the total number of infected.

Model parameters π_1, π_2, π_3 were computed by solving nonlinear equations,

$$\frac{\pi_1 c_{ss}^2}{\pi_1 c_{ss}^2 + \pi_2 n_{ss}^2 + \pi_3} = 1/6$$

$$\frac{\pi_2 n_{ss}^2}{\pi_1 c_{ss}^2 + \pi_2 n_{ss}^2 + \pi_3} = 1/6 \qquad (6.11)$$

$$R_\infty + D_\infty = 1 - I_\infty = 0.4$$

We assumed that at the beginning of epidemic one third of virus transmission comes from economic activities: one sixth - from consumption and one sixth - from work. Additionally, we assumed that total number of infected people at the end of epidemic that are either recovered or dead is 40%.

Equations (6.11) were solved by a constrained optimization method: the lower bound of infection rate π_3 was equal to the sum of recovery and death rates, $\mu + \gamma$.

The calibrated values are: $\pi_1 = 1.5\ 10^{-7}, \pi_2 = 9.5\ 10^{-5}, \pi_3 = 0.5$. Model parameters are presented in Table.1.

Notation	Economic Interpretation	Parameter Value
β	Weekly discount factor	$0.98^{1/52} = 0.9996$
θ	Working hours multiplier in household utility function	0.19
θ_π	Taylor rule coefficient of inflation	1.5
θ_x	Taylor rule coefficient of output gap	0.5/52
ξ	Calvo price stickiness (weekly)	0.98
$\hat{\gamma}$	Price dispersion parameter	1.35
δ	Capital depreciation rate (weekly)	0.06/52
α	Marginal product of labor	2/3
γ	Weekly probability of dying	0.25%
μ	Weekly probability of recovering	$7/_{14} - \gamma = 49.8\%$
π_{ss}	Steady-state inflation	1
r_{ss}	Steady-state nominal interest rate	$1/_{\beta} = 1.0004$
y_{ss}	Weekly average income	$58{,}000/52 = 1{,}115\ \frac{5}{13}$
n_{ss}	Steady-state number of work hours per week	28
A	Cobb-Douglass production function multiplier	$\frac{\beta(1-\alpha)}{1-\beta(1-\delta)}\left(\frac{y_{ss}}{n_{ss}}\right)^2 = 2.148$

Table.1. ERT model parameters. Model time frequency is weekly.

VII. Modelling Economic Effects of Epidemic

In this section we present simulations results obtained with the aid of ERT[1] model. We assumed that at time zero there is 0.05% of infected population. Calculations were performed with Juillard et al. (1998) numerical algorithm applicable to non-linear models. This algorithm solves dynamic macroeconomic models with perfect foresight expectations of economic agents. Since numerical method can diverge when applied with final parameters, we solve this model by using homotopy method where we adjusted parameters incrementally step-by-step.

[1] Authors converted Dynare model file to "yaml" format. The original ERT model code is available at: https://sites.google.com/site/mathiastrabandt/home/research

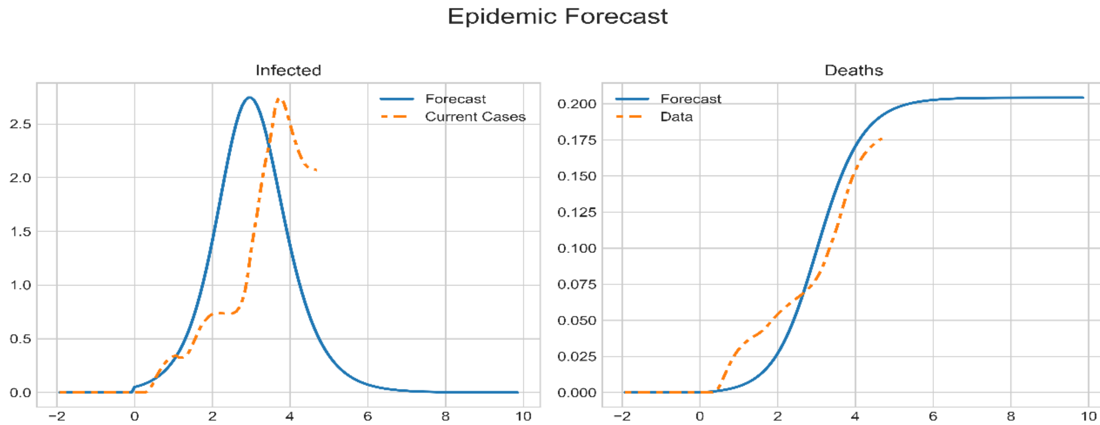

Fig.7.1. Forecast of infected and deceased. Blue line shows ERT model predictions and orange line shows the US data. We assumed that epidemic started on March 1st, which coresponds to time zero. X axis shows time in quarters and Y axis shows percentage of population.

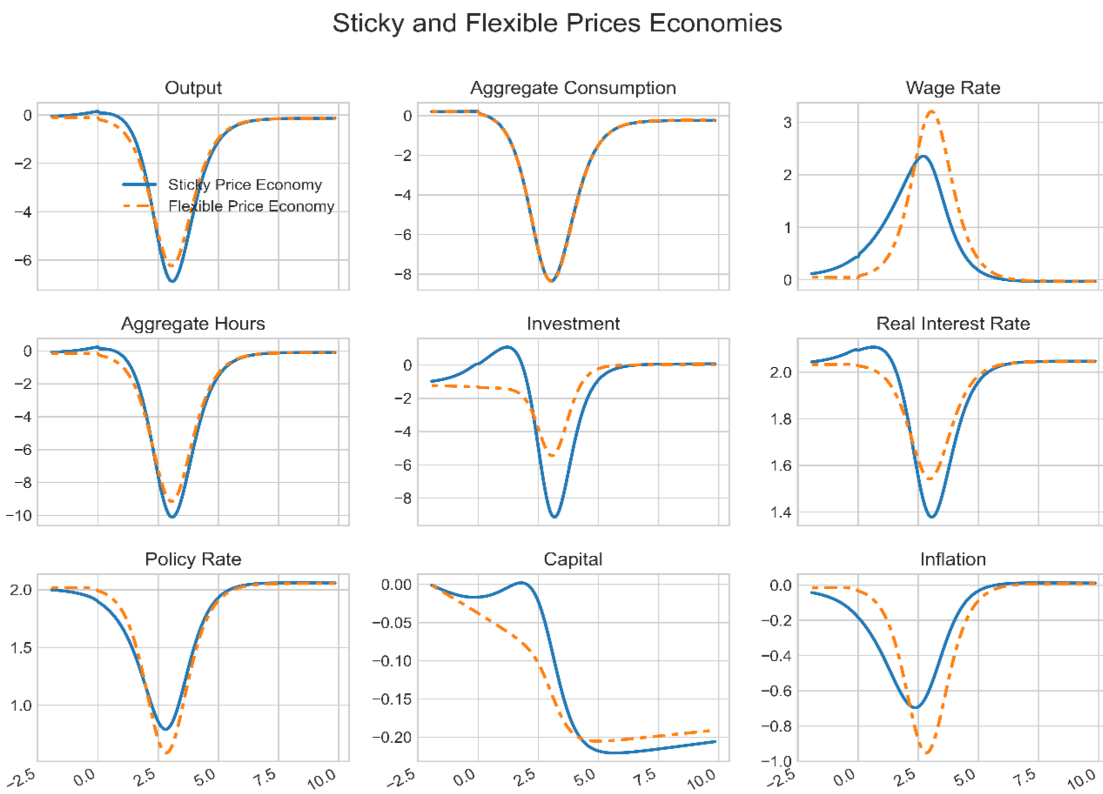

Fig.7.2. Macroeconomic variables are shown as percentage deviations from their initial steady state. The suspected, infected, recovered and deceased are shown in percent of initial population. The blue color lines mark plots of macroeconomic variables for sticky price economy ($\xi = 0.98$), and the dotted red lines, for flexible price economy ($\xi = 0$).

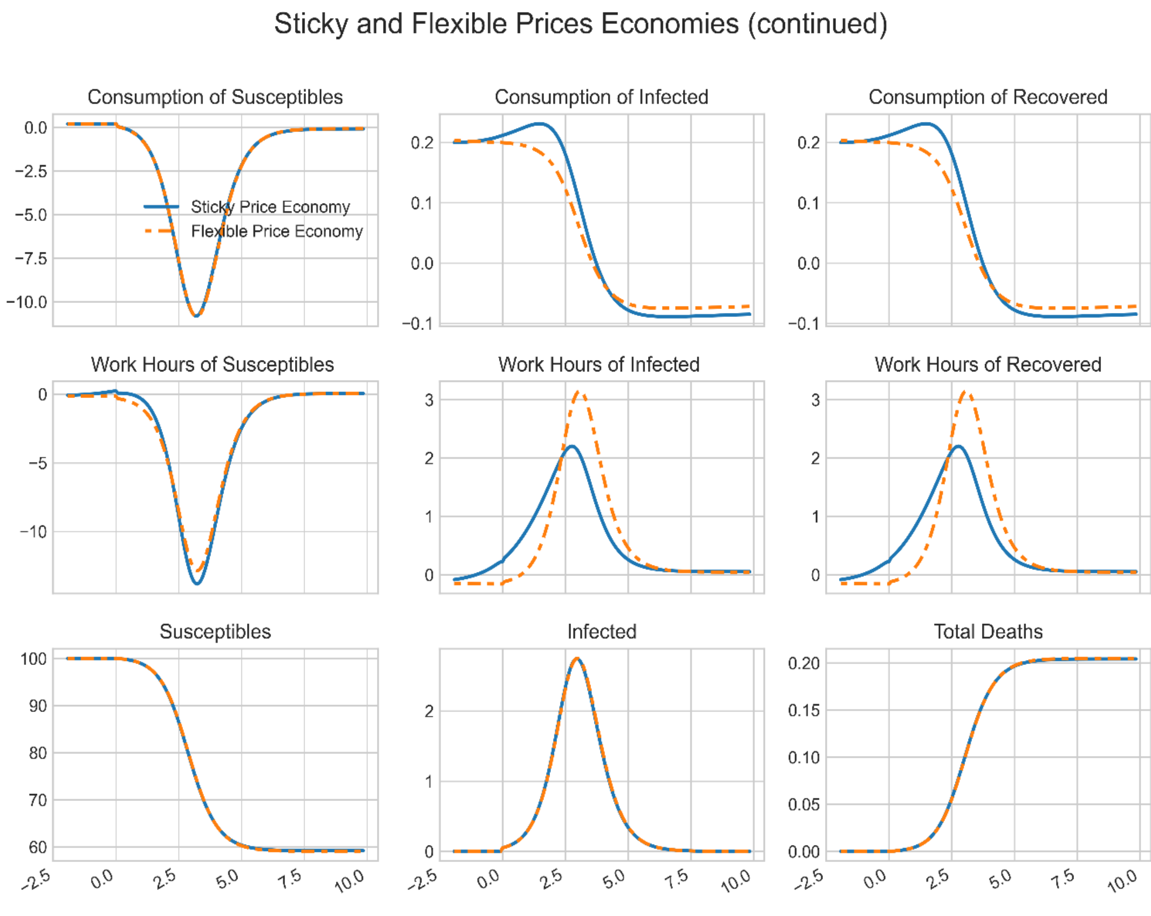

Fig.7.3. Macroeconomic variables are shown as percentage deviations from their initial steady state. The blue color lines mark plots of macroeconomic variables for sticky price economy ($\xi = 0.98$), and the dotted red lines – for flexible price economy ($\xi = 0$).

The basic reproduction number can be computed based on rate of infection transmission π_3 , and on weekly probabilities of recovery π_R and deaths π_D ; it is $R_b = \frac{\pi_3}{\pi_R+\pi_D} = 1$. As we have seen before, epidemiological model predicts no outbreak of disease in this case. However, because of agent's economic activities, this reproduction number becomes larger: $R_b = \frac{\pi_1 c_{ss}^2+\pi_2 n_{ss}^2+\pi_3}{\pi_R+\pi_D} = 1.3$. This warrants onset of epidemic. The computed peak number of infected is 2.8%. The total number of suspected and recovered reaches 59% and 41% limit, and the total number of deaths - 0.2%.

Epidemic negatively affects economy: output drops by staggering 6.9%, aggregate consumption, by 8.3% and aggregate work hours, by 10.1%. Susceptible population experiences the most significant drop compared to infected and recovered.

VIII. Modelling Lockdown Effects

Lockdown policies such as "stay-at-home", "shelter-in-place", closure of restaurants and gyms, prohibition of public gathering, etc., are restrictive policies announced by government to save lives during pandemic. These policies are detrimental and costly in terms of economic activity but beneficial in terms of public health.

The number of newly infected is decreased by a lockdown,

$$T_t = \{\pi_1 (S_t c_t^S)(I_t c_t^I) + \pi_2 (S_t n_t^S)(I_t n_t^I) + \pi_3 (S_t I_t)\} (1 - \vartheta L_t)^2 \quad (8.1)$$

Here parameter ϑ is the lockdown intensity, and L_t is the time dependent lockdown policy. Lockdown decreases infection transmission from infected to suspected. We assume a quadratic dependence of infection transmission on lockdown, $(1 - \vartheta L_t)^2$. Similarly, work hours of susceptible and infected decrease by a factor of $(1 - \vartheta L_t)$.

Caselli et al. (2020) show that full lockdown suppresses mobility by maximum of 30%, social distancing by 2%, and jobs posting by 10%. Considering these numbers, we estimate the average effectiveness of lockdown as 10%, or $\vartheta = 0.1$. Below we present simulation results of a scenario, where lockdown occurs at month four and lasts for three quarters. The gray shaded area in figures 10.2-3 shows occurrence of this lockdown.

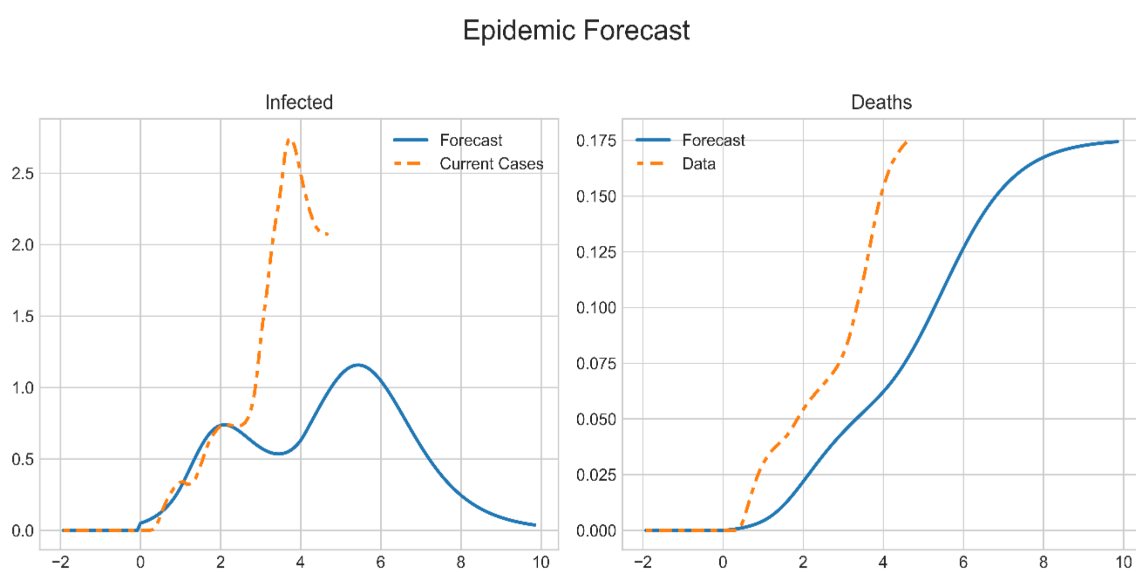

Fig.8.1. Forecast of infected and deceased. Blue color line shows ERT model predictions and orange color line, the US data. Time zero corresponds to the start of epidemic on March 1st. X axis displayes time in quarters, and Y axis shows percentage of population.

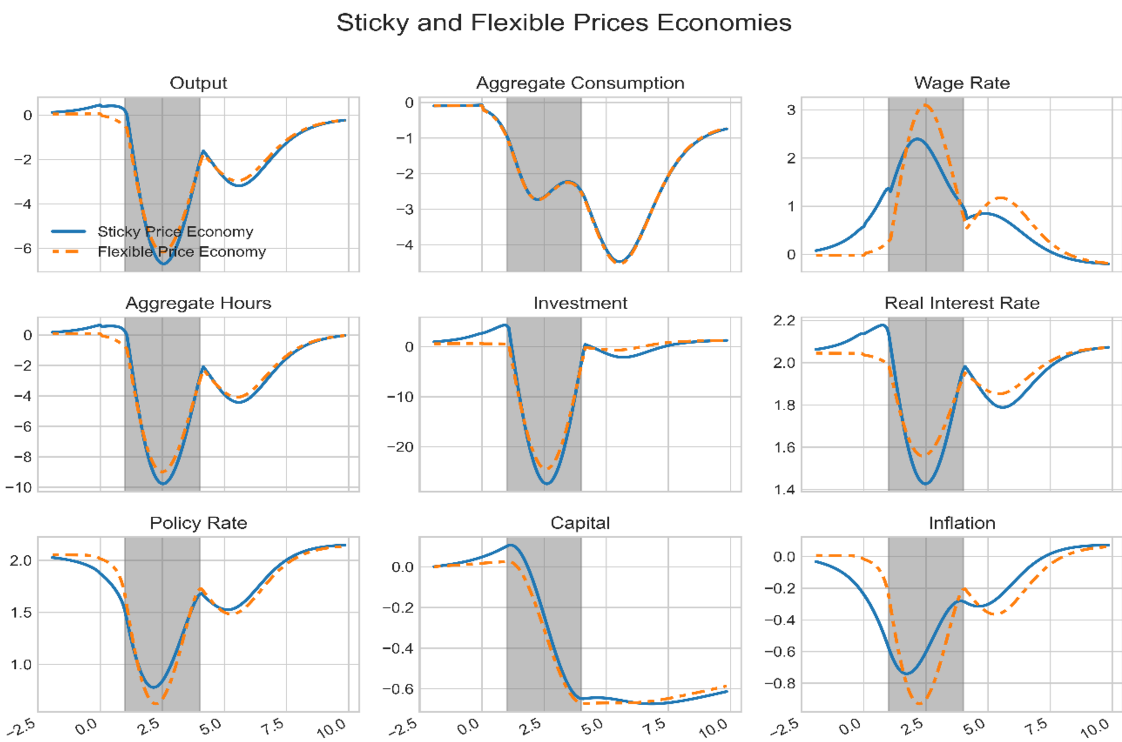

Fig.8.2. Effects of lockdown on economy. Macroeconomic variables are shown as percentage deviations from their initial steady state. The gray shaded area displays occurrence of lockdown.

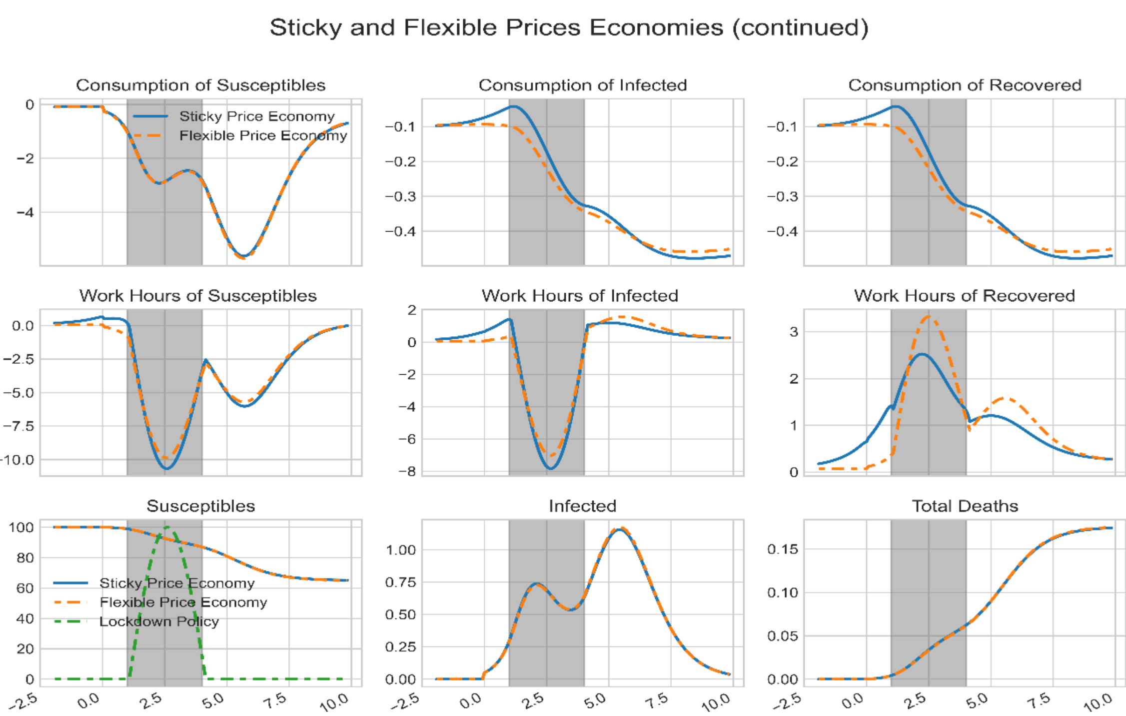

Fig.8.3. Effects of lockdown on economy.

These graphs illustrate, that if a lockdown is lifted prematurely, infection could resume its course. Lockdown policy is beneficial in terms of reduction of virus spreading. For example, the number of infected at peak drops by a factor of two, while cases of deaths decrease by 0.03%. Consumption drops to -4.5% for lockdown policy compared to -8.3% when there is no lockdown. Lockdown impact on output is twofold: on the one hand, the number of infected decreases and output increases; on the other hand, the work hours of individuals reduce and the output decreases. These two effects offset each other, and maximum drop in output does not change significantly. Lockdown is lifted at the end of fourth quarter and infection transmission resumes. This results in additional output drop at quarter six. We may conclude that lockdown policies are efficient in the short run but are not efficient in the long run. Contrary to common belief, tighter lockdown policy today could lead to an increase of the number of infected in the future.

We repeated calculations for lockdown intensity of 5%. With easy policy the second wave of infection does not emerge. The number of infected and decease cases are larger compared to the previous scenario. However, the number of deaths at period ten is about the same.

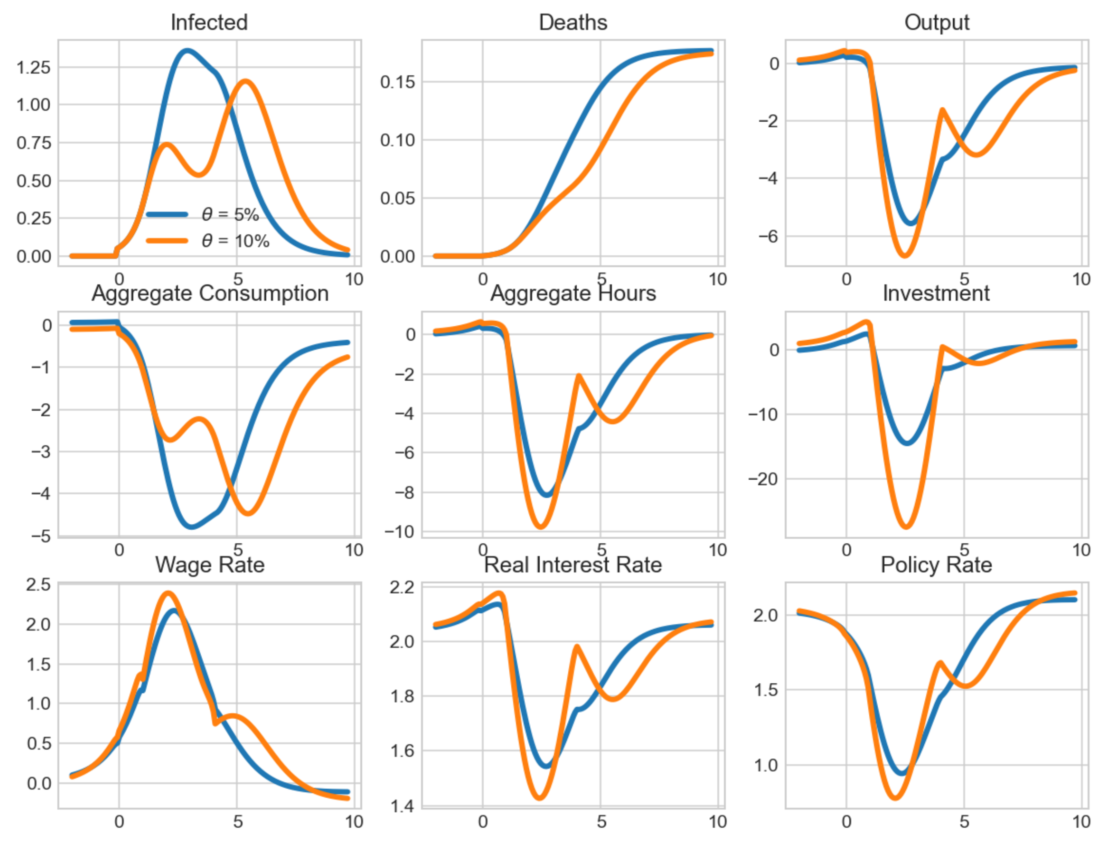

Fig.8.4. Forecast of lockdown scenarios with intensities of 5% and 10% marked by blue and orange color lines, respectively. Vertical axis displays percentage deviation from initial steady state.

We repeated calculations where we increased lockdown intensity from 0 to 20%. For each scenario we computed total production loss and maximum drop in production. Total loss was computed as an area under the output curve. Figure 10.5 illustrates increase of total loss with growth of lockdown intensity. However, the output value at trough firstly increases with lockdown intensity and secondly, decreases. This could be explained by widening of output curve. This results in U shape type recovery.

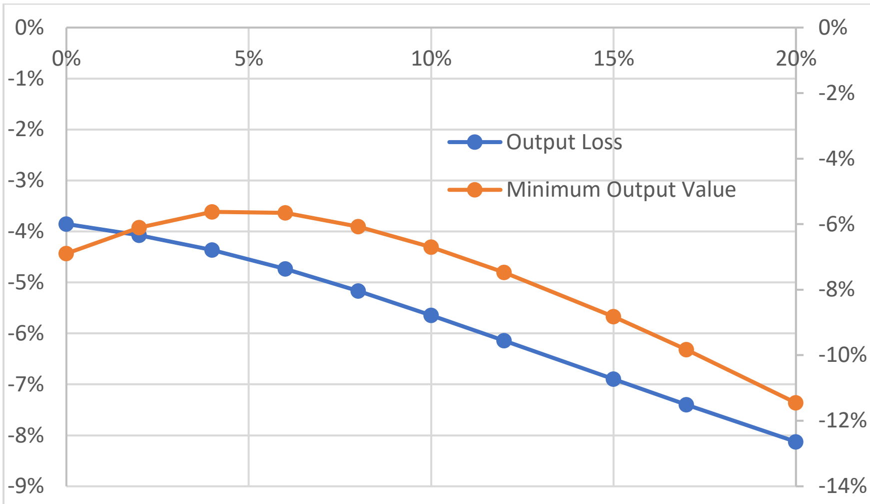

Fig.8.5. Total production loss as a function of lockdown intensity ϑ. Left vertical axis shows values of output loss, and right axis – values of output at trough.

This figure illustrates that a stringent lockdown policy negatively affects country's production.

IX. Vaccination

The spread of COVID-19 diseases could be eradicated by establishment of a mass vaccination program. The vaccine helps individuals to develop antibodies and become immune to this disease. We assume that vaccination rate of individuals is constant, ρ. The number of vaccinated individuals during a period of dt is $\rho S_t V_t dt$, where V_t is the vaccination policy. Because of vaccination, the number of newly infected decreases while the number of immune to this disease increases:

$$\begin{aligned}
\frac{dS}{dt} &= -\beta IS - \rho SV \\
\frac{dI}{dt} &= -\beta IS - (\mu + \gamma) I \\
\frac{dR}{dt} &= \mu I + \rho SV \\
\frac{dD}{dt} &= -\gamma I
\end{aligned} \tag{9.1}$$

Below we present results of simulations for $\rho = 0.02$ and duration of vaccination of three quarters. Vaccination of 2% per week means that all population would be vaccinated in one year, assuming the stock of susceptible had not decreased with time. We assumed that vaccination program starts at the second quarter since the epidemic outbreak. Vaccination helps to reduce number of infected people and to improve economic outlook: the number of infected population at peak drops by a factor of four and reaches at peak 0.6% compared to 2.7%, while the output drops to -1.9% compared to -6.9% in the absence of this vaccination program.

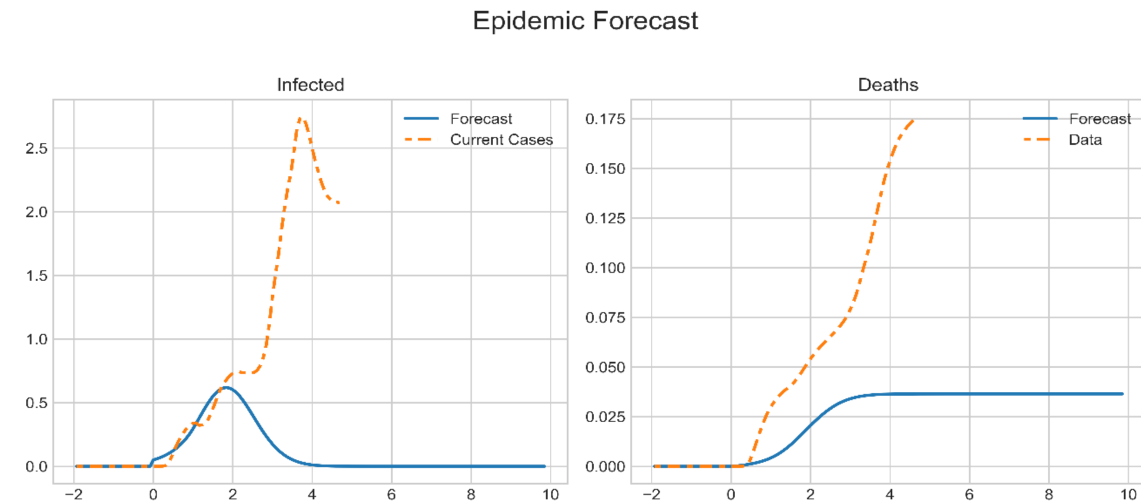

Fig.9.1. Forecast of infected and deceased. Blue color lines show ERT model predictions and orange color lines show number of current cases of infected and total number of deaths in the US. We assumed that epidemic started on March 1st , which coresponds to time zero. X axis shows time in quarters and Y axis shows percentage of population.

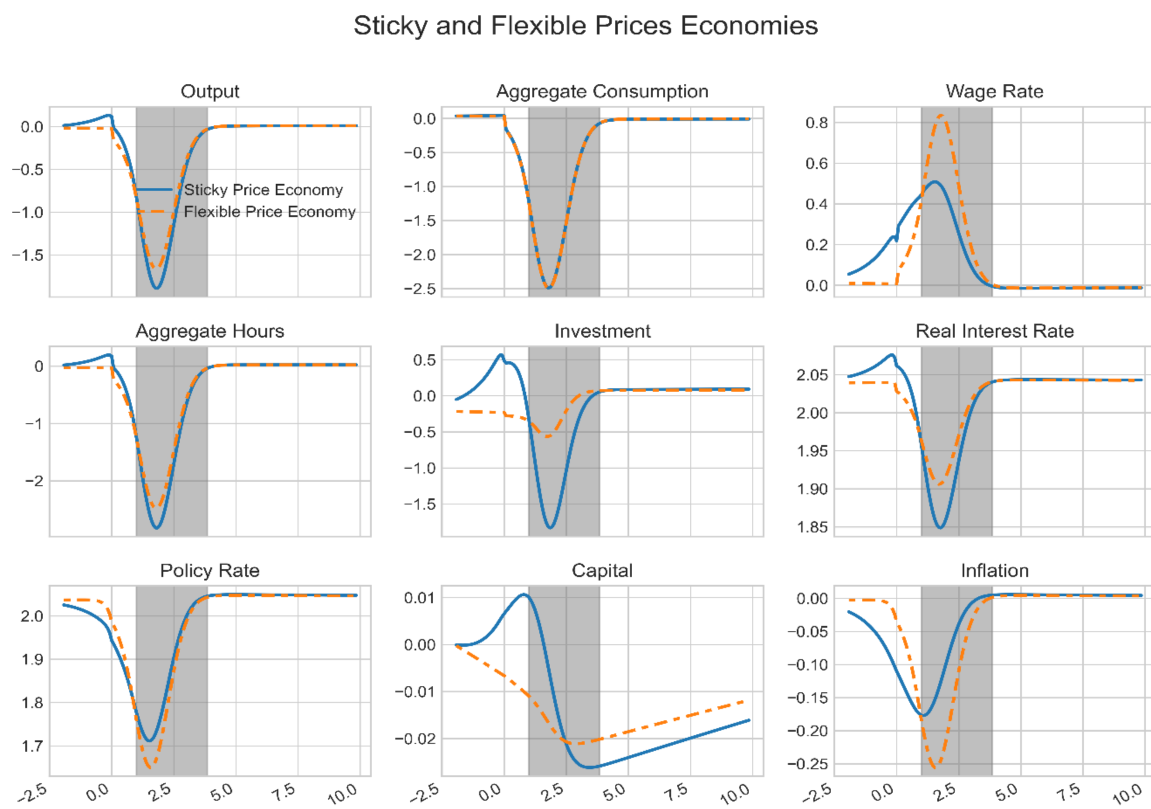

Fig.9.2. Macroeconomic variables are shown as percentage deviations from their initial steady state.

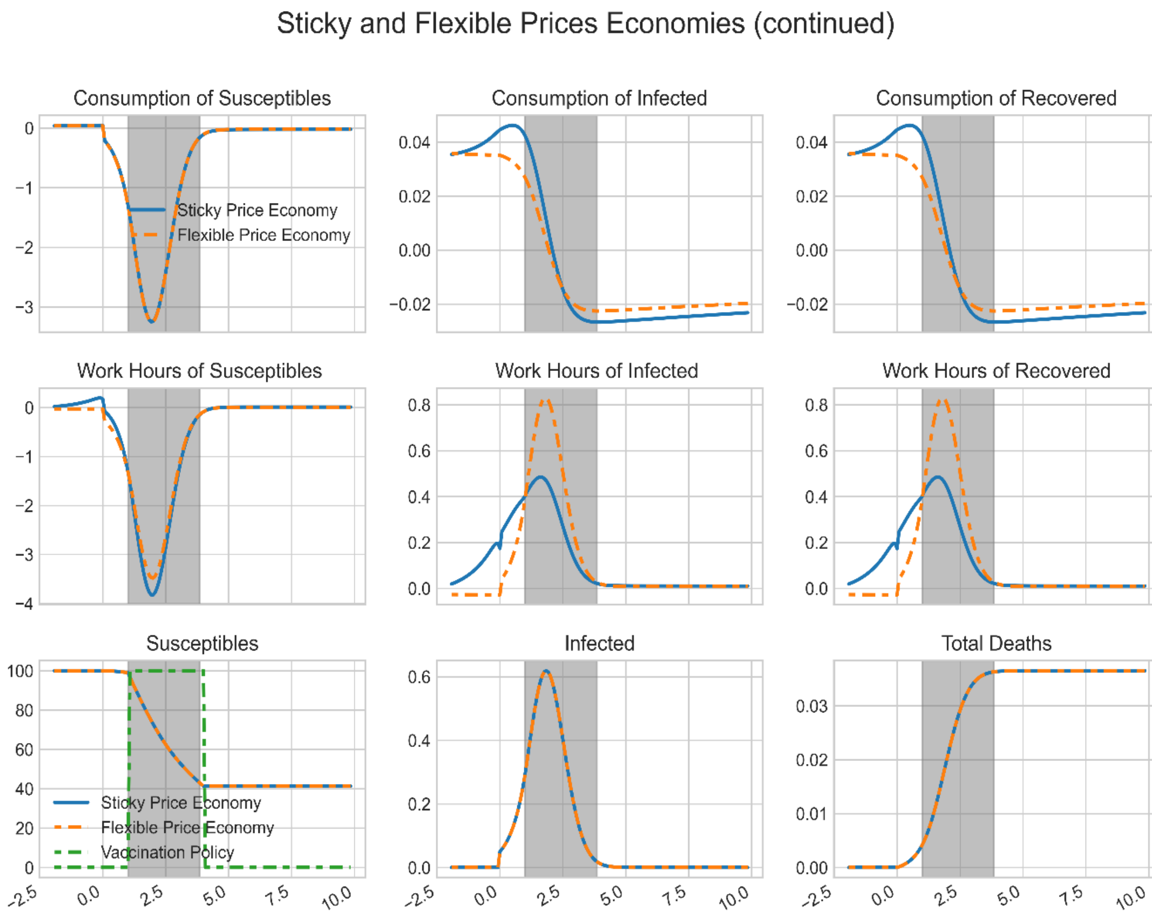

Fig.9.3. Macroeconomic variables are shown as percentage deviations from their initial steady state. Suspected, infected, recovered and deceased are in percent of initial population. The gray shaded area displays occurrence of vaccination program.

Vaccination also improves economic outlook in terms of working hours of susceptible: -3.8% versus -13.8%, and in terms of consumption: -3.2% versus -10.8%.

X. Mitigation Policies

In this section we study effects of vaccination and lockdown policies on economy. To account for economic cost of lives loss, we computed the discounted future income of an individual for a duration of 20 years. We assume that 20 years is the average stint of individual's employment. We then subtracted this value of future income from the value of utility (6.5).

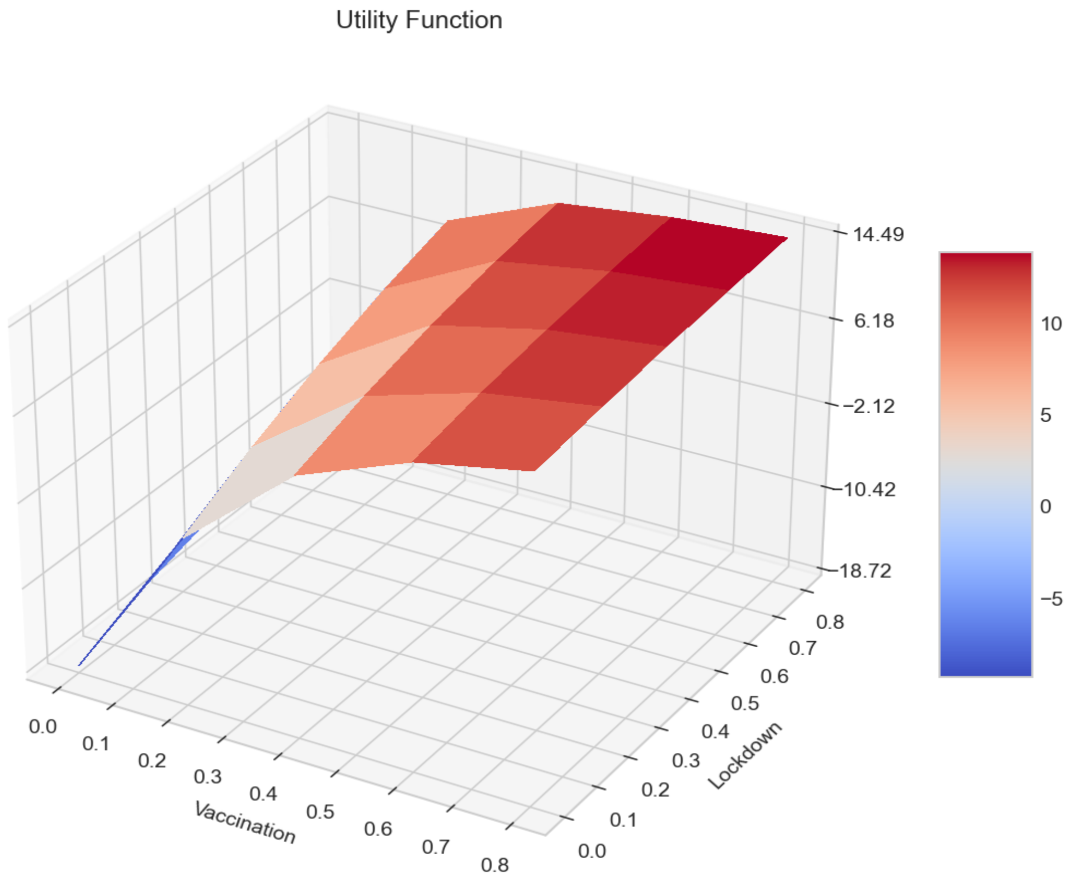

Fig.10.1. Utility function versus intensity of lockdown and vaccination policies. Lockdown and vaccination start at the second quarter since epidemic outbreak and last for three quarters. Maximum lockdown and vaccination intensities were 10% and 2%, respectively. X and Y axis display multiplier factors, where factor of one corresponds to maximum policy intensity.

Total utility increases with tightening of lockdown and vaccination policies. It is worth noting, that this function is more sensitive to vaccination than to lockdown intensity.

Figure 10.2 presents comparison of macroeconomic forecast of lockdown and vaccination policies. Lockdown intensity was 10%. Vaccination program is beneficial in terms of saved lives and in terms of economic outlook - the number of deaths is reduced by 0.16%, which is equivalent to half a million lives in the US. Lockdown policies are helping to reduce number of infected and deceased as well. However, lockdown has a negative impact on country economic outlook including its output, investment, and individuals' consumption.

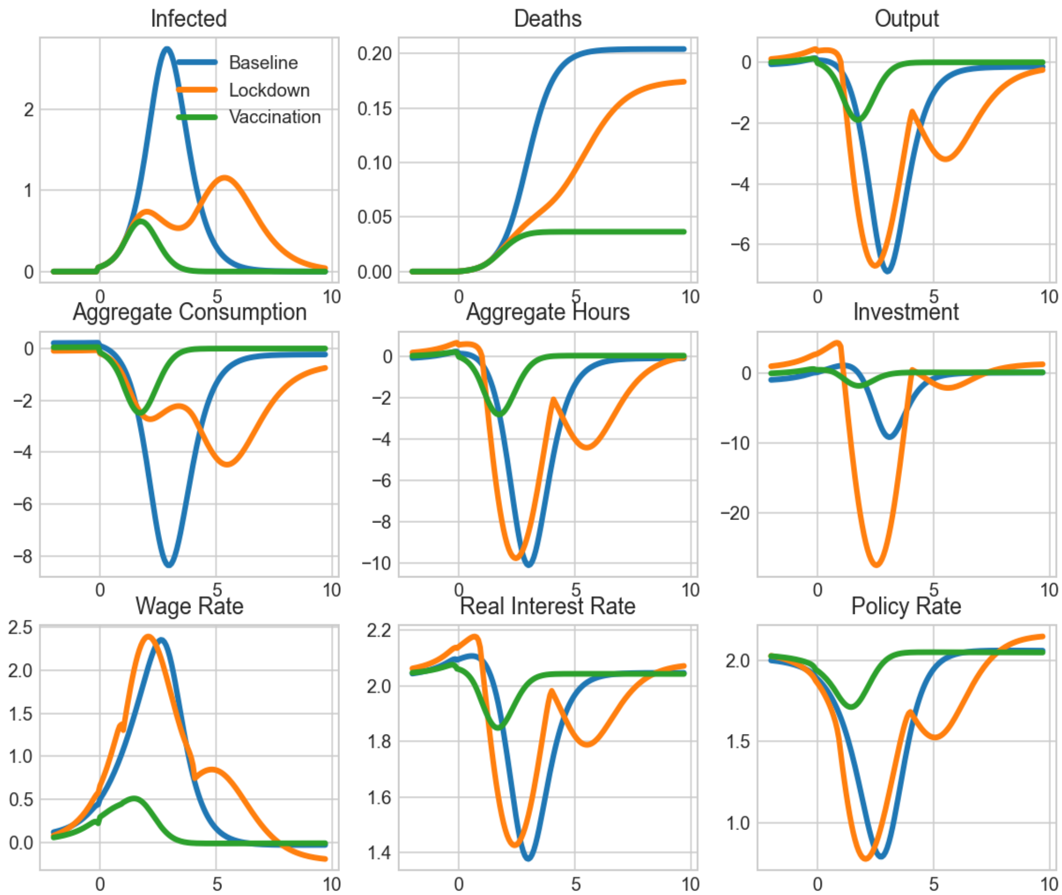

Fig.10.2. Blue color lines mark baseline scenario, and orange and green color lines mark lockdown and vaccination scenarios, respectively. Vertical axis displays percentage deviations of macroeconomic variables from their initial steady state.

For illustration purposes, we present results when both policies are in place. Vaccination and lockdown rates were 3% and 5%, respectively. Weekly vaccination rate of 3% means that in four months half of a population will be vaccinated. This estimate is consistent with percentage of vaccinated individuals in the US by the end of May 2021. Those are individuals who have received at least one dose of COIVID-19 vaccine.

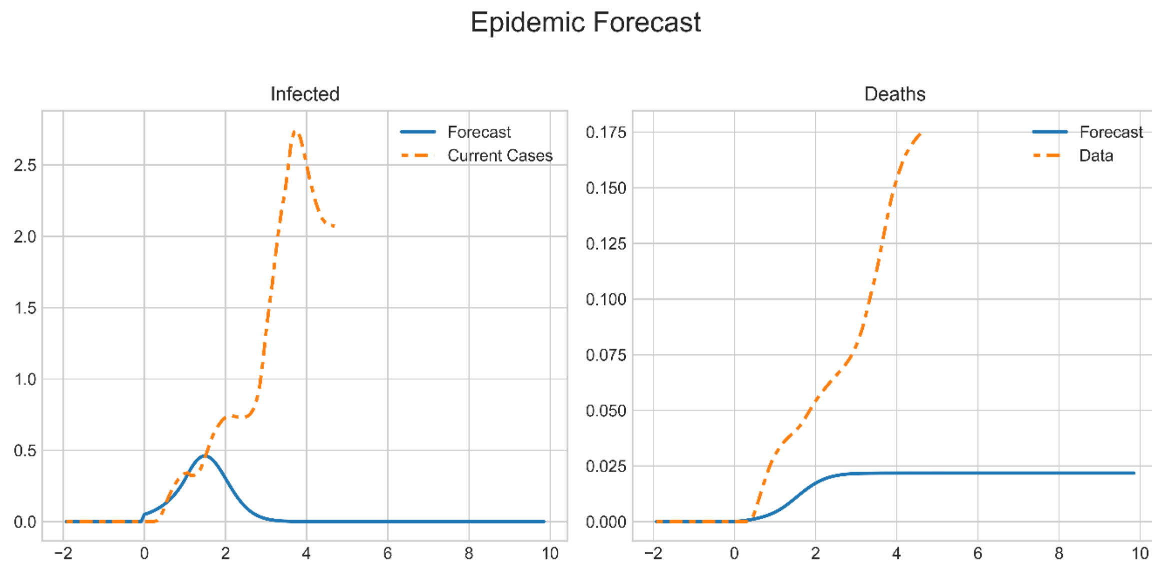

Fig.10.3. Forecast of infected and deceased. Blue color lines show ERT mode predictions and orange lines show current cases of infected and total number of deaths in the US. Lockdown intensity was 5% and vaccination rate was 3%.

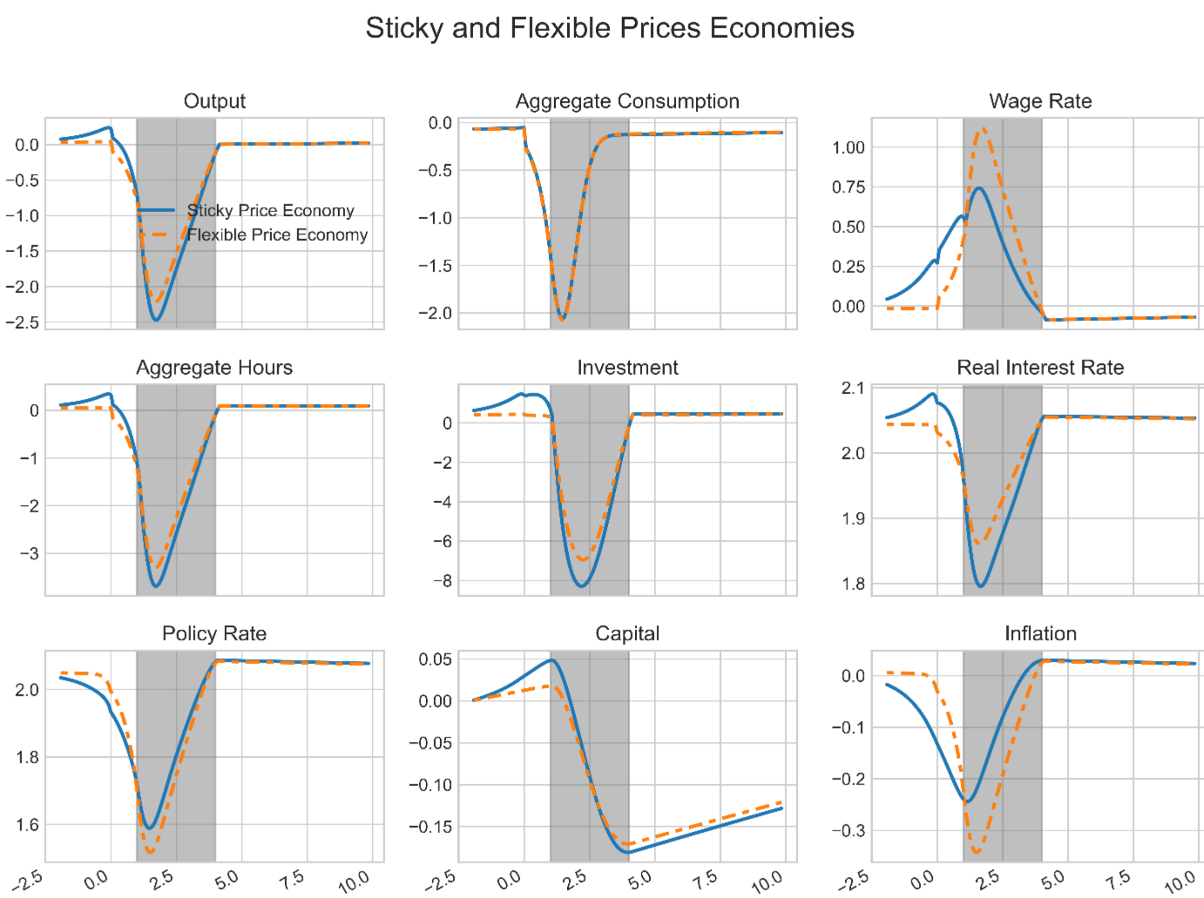

Fig.10.4. Macroeconomic variables are shown as percentage deviations from their initial steady state. Gray shaded area shows occurrence of vaccination and lockdown programs, which start at the second quarter since the start of epidemic and last for three quarters. Lockdown and vaccination rates were 5% and 3%, respectively.

The path of economic recovery has a V shape, and impact of epidemic is much milder.

In what follows, we consider a vaccination program only. We repeated these simulations with the only difference that vaccination program starts at different times. We observed that timing of adopting vaccination program plays a crucial role on epidemic transmission. We simulated three scenarios, where vaccination program lasts for three quarters and starts at the second, the fourth and the sixth month since epidemic outbreak. Starting program earlier than later improves output and aggregate consumption outlook: -0.7% versus -1.9% and -4.3%, and -1% versus -2.5% and -5.3%, respectively.

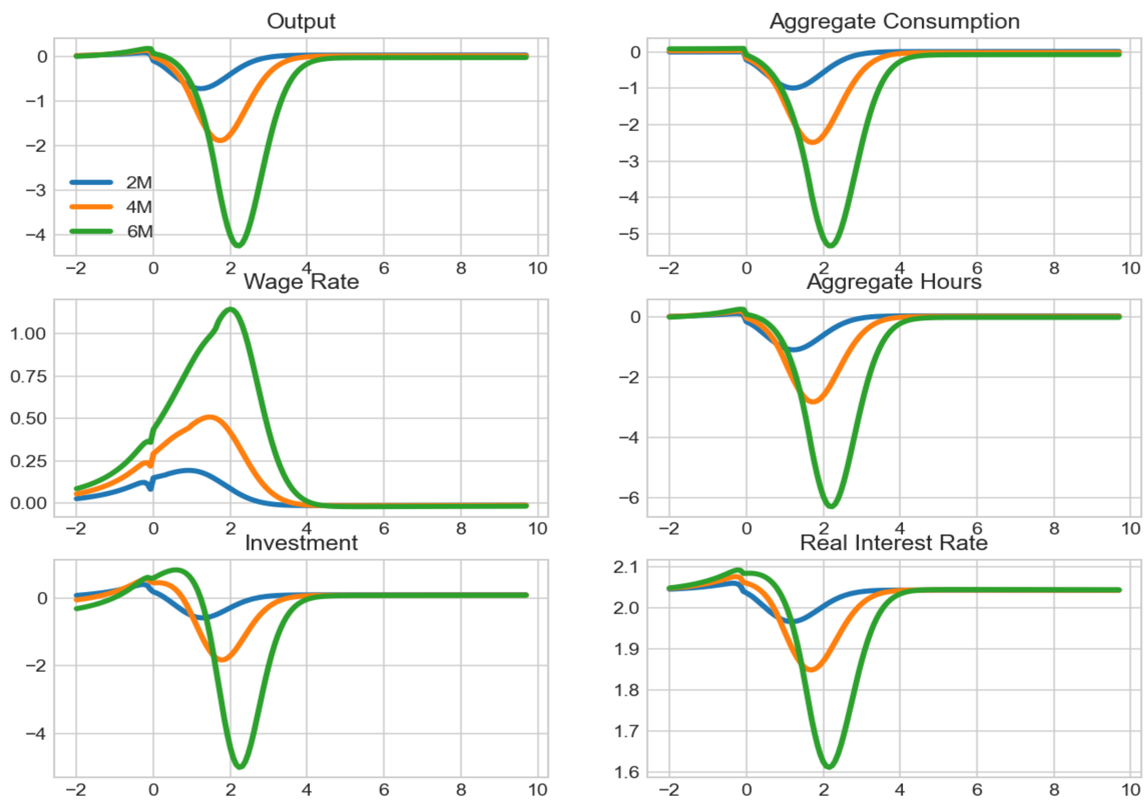

Fig.10.5. Macroeconomic variables are shown as percentage deviations from their initial steady state. The blue, orange, and green colors mark lines where vaccination starts at the second, the fourth and the sixth month since the start of epidemic. Starting program later reduces benefits of vaccination program.

All in all, we can conclude that vaccination is the most effective mean to combat detrimental effects of virus.

XI. GSW MODEL

ERT model lacks a reference to unemployment - instead, it predicts working hours of individuals. The theory of unemployment was developed by Gali (2011 a, b). Later it was embedded into the new Keynesian model framework of Gali, Smets and Wouters (2012). The following equations constitute GSW model:

Consumption Euler equation,

$$c_t = C_1 c_{t-1} + (1 - C_1)E(c_{t+1}) - C_2\big(r_t - E(\pi_{t+1}) + \varepsilon_t^b\big) \quad (11.1)$$

Here $C_1 = \frac{h/\tau}{1+h/\tau}$; $C_2 = \frac{1-h/\tau}{1+h/\tau}$ are the constants, h is the external habit parameter, τ is the trend growth rate, r_t is the nominal interest rate and ε_t^b is the exogenous AR(1) risk premium process. Equation (11.1) illustrates that consumption today depends on its historical value, as well as on its future expectations.

Investment follows Euler equation,

$$i_t = I_1 i_{t-1} + (1 - I_1)E(i_{t+1}) + I_2 q_t + \varepsilon_t^q \quad (11.2)$$

Where $I_1 = \frac{1}{1+\beta}$; $I_2 = \frac{I_1}{\tau^2\Psi}$ are the constants, β is the household's discount factor, Ψ is the elasticity of capital adjustments cost, q_t is the value of installed capital, and ε_t^q is the exogenous AR(1) process of investment technology shock.

Investment depends on the value of capital stock,

$$q_t = -\big(r_t - E(\pi_{t+1}) + \varepsilon_t^b\big) + Q_1 E\big(r_{t+1}^k\big) + (1 - Q_1)E(q_{t+1}) \quad (11.3)$$

Here $Q_1 = \frac{r^k}{r^k+1-\delta}$ is the constant. Goods market clearing implies,

$$y_t = C_y c_t + I_y i_t + V_y v_t = M_p[\alpha k_t + (1-\alpha)n_t + \varepsilon_t^a] \quad (11.4)$$

Here $C_y = {}^{c_{ss}}/_{y_{ss}}$; $I_y = {}^{i_{ss}}/_{y_{ss}}$; $V_y = {}^{r_{ss}^k k_{ss}}/_{y_{ss}}$ are the constants, v_t is the capital utilization rate, and M_p is the price markup.

Price-setting under the Calvo model is,

$$\pi_t^p - \gamma_p \pi_{t-1}^p = \beta\big[E\big(\pi_{t+1}^p\big) - \gamma_p \pi_t^p\big] - \frac{(1-\beta\theta_\pi)(1-\theta_\pi)}{\theta_\pi\big(1+(M_p-1)\zeta_p\big)}\big(\mu_{p,t} - \mu_{p,t}^n\big) \quad (11.5)$$

Here ζ_p is the curvature of the Dixit-Stiglitz (1977) price aggregator.

Average and natural price markups are described as,

$$\mu_{p,t} = -(1-\alpha)\,\omega_t - \alpha r_t^k + \varepsilon_t^a \quad (11.6)$$
$$\mu_{p,t}^n = 100\,\varepsilon_t^p$$

Here $\omega_t = w_t - p_t$ is the real wage. Wage-setting under the Calvo model is,

$$\pi_t^w - \gamma_w \pi_{t-1}^p = \beta\left[E(\pi_{t+1}^w) - \gamma_w \pi_t^p\right] - \frac{(1-\beta\theta_w)(1-\theta_w)}{\theta_w(1+\varepsilon_w\varphi)}\left(\mu_{w,t} - \mu_{w,t}^n\right) \quad (11.7)$$

Average and natural wage markups, and unemployment equations are:

$$\mu_{w,t} = \omega_t - \left(z_t + \varphi n_t + \varepsilon_t^\chi\right) = \varphi u_t$$
$$\quad (11.8)$$
$$\mu_{w,t}^n = 100\,\varepsilon_t^w = \varphi u_t^n$$

Where ε_t^χ is the labor supply shock. The trend for aggregate consumption is,

$$z_t = (1-\upsilon)z_{t-1} + \upsilon\left[\frac{1}{1-h/\gamma}\,c_t - \frac{h/\gamma}{1-h/\gamma}\,c_{t-1}\right] \quad (11.9)$$

Labor force is composed of individuals work hours and involuntarily unemployed individuals' hours:

$$l_t = n_t + u_t \quad (11.10)$$

Inflation rate, by definition, is equal to minus involuntarily unemployed individuals' hours, -u_t. Capital accumulation is described by equation:

$$k_t - \upsilon_t = \left(1 - \frac{i_{ss}}{k_{ss}}\right)(k_{t-1} - \upsilon_{t-1}) + \frac{i_{ss}}{k_{ss}}\,i_t + \frac{i_{ss}}{k_{ss}}\,(1+\beta)\,\tau^2\Psi\,\varepsilon_t^q \quad (11.11)$$

The optimal capital utilization condition reads,

$$\upsilon_t = \frac{1-\psi}{\psi}\,r_t^k \quad (11.12)$$

Here ψ is the elasticity of capital utilization cost function. The optimal input choice is given by,

$$k_t = \omega_t - r_t^k + n_t \quad (11.13)$$

Lastly, monetary policy rule for short term nominal interest rate is,

$$r_t = \rho_r\,r_{t-1} + (1-\rho_r)\left[r_\pi \pi_t^p + r_y(y_t - y_t^n) + r_{\Delta y}\{(y_t - y_t^n) - (y_{t-1} - y_{t-1}^n)\}\right] \quad (11.14)$$

Exogeneous disturbances for productivity ε_t^a, risk premium ε_t^b, investment specific technology ε_t^q, and government expenditure ε_t^g follow AR(1) process with IID-Normal innovation terms η_t:

$$\begin{aligned}
\varepsilon_t^a &= \rho_a \varepsilon_{t-1}^a + \eta_t^a \\
\varepsilon_t^b &= \rho_b \varepsilon_{t-1}^b + \eta_t^b \\
\varepsilon_t^q &= \rho_q \varepsilon_{t-1}^q + \eta_t^q \\
\varepsilon_t^g &= \rho_g \varepsilon_{t-1}^g + \eta_t^g + c_{gy}\, \varepsilon_t^a
\end{aligned} \tag{11.15}$$

Table 2 presents calibration parameters of GSW model. These parameters were estimated by Mihailov (2020). GSW model was calibrated based on US quarterly data for period from 1999Q1 to 2017Q4. These time series included data for real GDP, its deflator, real consumption, interest rate on lending facilities, employment, and unemployment rates. We corrected investment adjustment cost to 0.2 from its original value of 3.96 to better account for year 2020 output decline.

Notation	Economic Interpretation	Value
Ψ	Elasticity of capital adjustment cost	0.2
h	External habit	0.75
φ	Inverse Frisch elasticity of labor supply	4.35
υ	Short-term wealth effect on labor supply	0.58
θ_p	Calvo price stickiness	0.62
θ_w	Calvo wage stickiness	0.55
γ_p	Price indexation	0.49
γ_w	Wage indexation	0.18
ψ	Capital utilization	0.56
M_p	Gross price markup	1.5
ρ_r	Interest-rate smoothing	0.86
r_π	Policy feedback to inflation	1.89
r_y	Policy feedback to output gap	0.16
$r_{\Delta y}$	Policy feedback to change in output gap	0.25
$10^2(\beta^{-1}-1)$	Steady-state time discount factor	0.31
τ	Trend growth rate	1.004
α	Elasticity of capital in production function	0.17
ρ_a	Neutral technology (TFP)	0.98
ρ_b	Risk premium	0.42
ρ_g	Aggregate net spending	0.97
ρ_q	Investment-specific technology	0.75
c_{gy}	Government expenditure shock coefficient	0.51

Table.2. GSW model calibration parameters. Model time frequency is quarterly.

A. Forecast

GSW model[1] equations are log linearized. To solve this model equations, we applied a perfect foresight algorithm. Adverse shocks to labor supply occur at the first quarter and last for one or two quarters. The value of these shocks was chosen to make labor supply drop by about 20%.

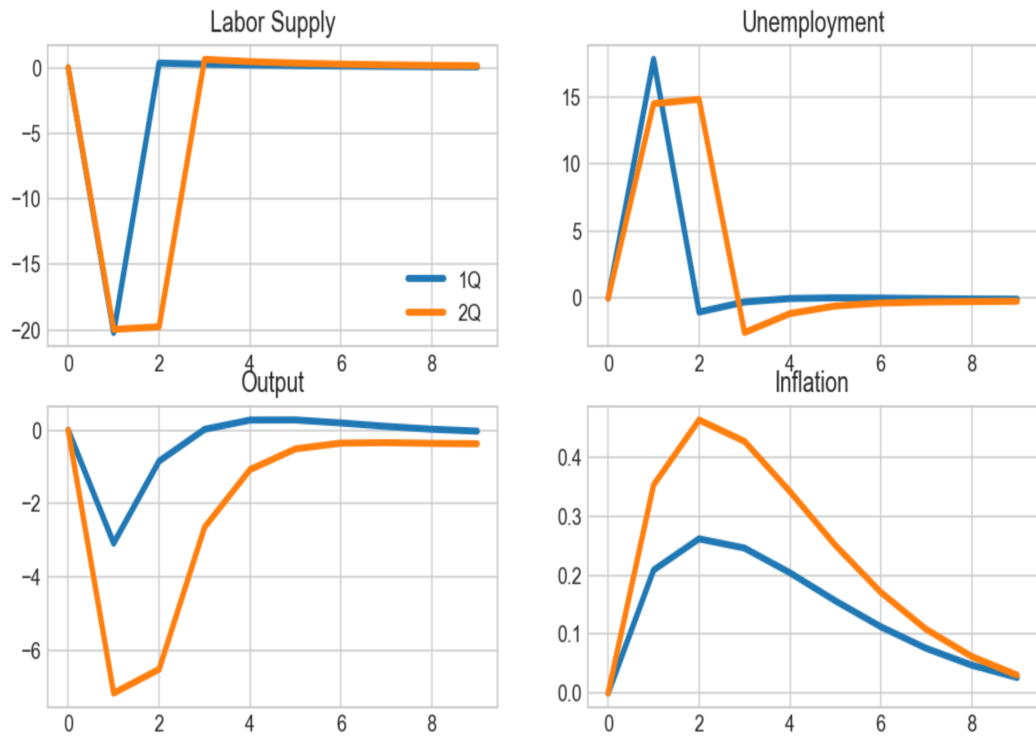

Fig.11.1. Forecast of effects of an adverse shock to labor supply on macroeconomic variables. Shocks occur at the first quarter and last for one or two quarters. Blue and orange color lines show response of macroeconomic variables to these shocks.

Graphs 11.2-3 present decomposition of deviations of output, work hours, investment, and inflation rate from its balanced path. Python software analyzes structure of equations and computes contribution of each endogenous variable and exogenous shock. Those variables and shocks are imposed one at a time to infer their contribution to macroeconomic variables paths.

[1] GSW model Dynare code was kindly provided by Alexander Mihailov in a zip archive via e-mail on December 8th, 2020. The Dynare model file was then translated to "yaml" format by authors.

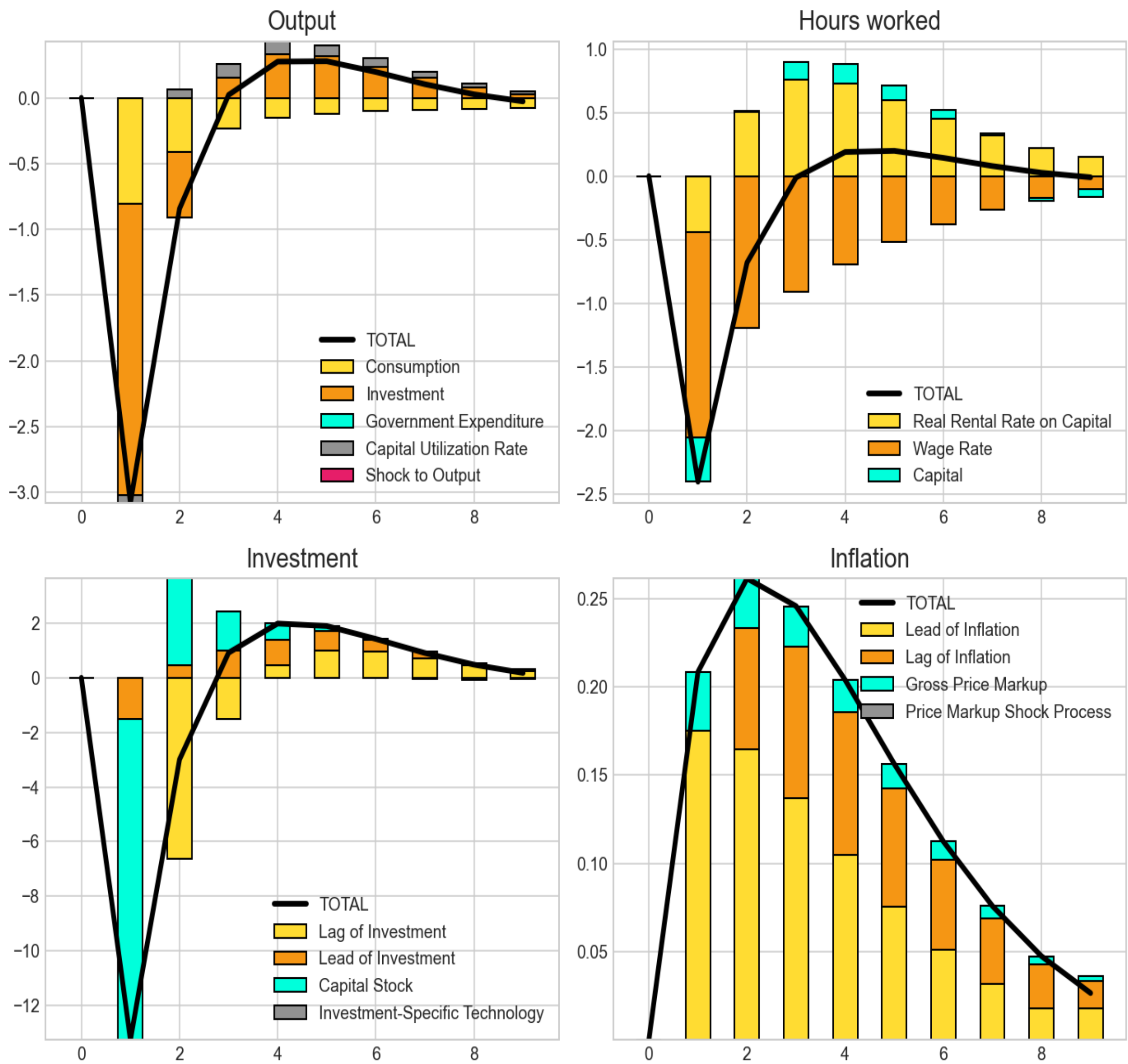

Fig.11.2. Forecast of an adverse labor supply shock to US economy. Shock occurs at the first quarter and lasts for one quarter. Vertical axis displays percentage deviation of macroeconomic variables from their balanced path.

To complete this picture, we show graphs of output, work hours, investment, and inflation for an adverse labor supply shock that lasts two quarters.

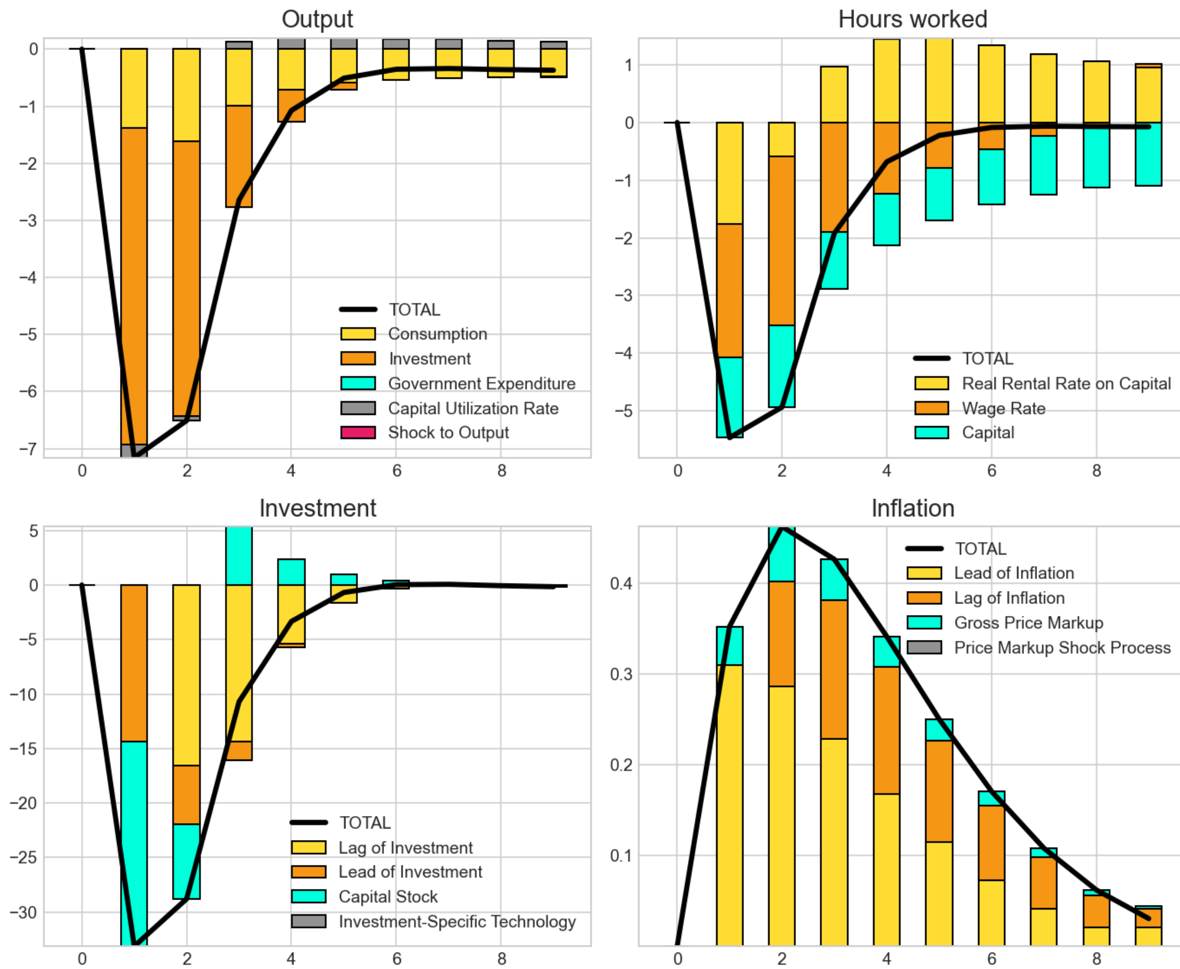

Fig.11.3. Forecast of an adverse labor supply shock to US economy. Shock to labor supply occurs at the first quarter and lasts for two quarters. Vertical axis displays percentage deviation of macroeconomic variables from their balanced path.

B. Judgmental Adjustments

In this section we present an example of user's judgmental adjustments to macroeconomic variables. Python platform allows one to set a specific path of one or more endogenous variables at different time periods. It is achieved by "exogenizing" endogenous variables and "endogenizing" shock variables. Economic agents anticipate or don't anticipate future shocks. We assume that agents make perfect foresight decisions and shocks are anticipated[1]. The values of these shocks are computed to bring the path of endogenous variables to the desired level.

[1] Assumption of anticipated shocks means that agents make rational expectation decisions on the best information available. For details on numerical algorithm please see Appendix C.

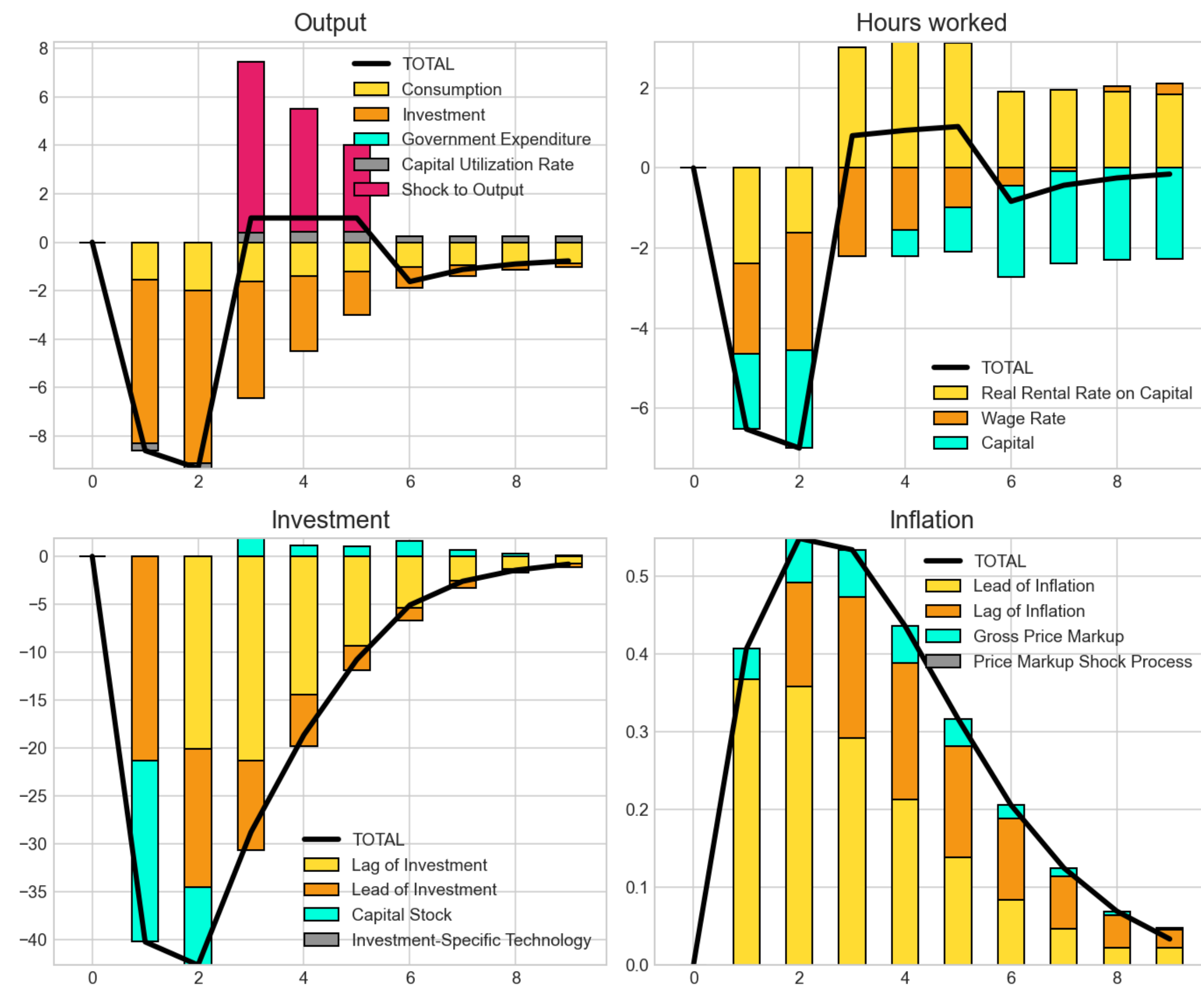

Fig.11.4. Forecast of macro variables with user's judgmental adjustments of output of 1% for the duration of three quarters. Adverse shock to labor supply occurs at the first quarter and lasts for two quarters. User makes a judgmental adjustment on output path. Vertical axis displays percentage deviation of macroeconomic variables from their balanced path.

Figure above shows forecast of macroeconomic variables with user's judgmental adjustments on output. User makes a judgement that output is 1% for three quarters starting at quarter three. The red color bars show values of output shocks which result in output path satisfying this judgmental adjustment.

XII. CONCLUDING REMARKS

We have developed a powerful and user-friendly platform for macroeconomic modeling in Python, including tools for filtering, simulation, estimation, forecasting and model diagnostics for Dynamic Stochastic General Equilibrium (DSGE) models. This platform can be applied for analysis of New Keynesian models, Real Business Cycle models, Gap models, and Overlapping Generations models, to name a few. This software is a quite versatile and a flexible toolbox.

For demonstration purposes we work with a non-linear stationary DSGE model to study macroeconomic effects of COVID-19 pandemic lockdown and vaccination policies. Our analysis utilizes Eichenbaum-Rebelo-Trabandt (2020) model which integrates the Neoclassical and the New Keynesian approaches with epidemiological concepts. The numerical calculations and the subsequent analysis are accomplished with the aid of this Python software. We study transmission of virus and its effects on country economy. We show that standard Susceptible-Infected-Recovered compartmental epidemiological model underpredicts rate of infection transmission and needs to account for virus transmission due to agents' economic activities. We consider several lockdown and vaccination policies and perform forecast simulations of ERT model with these policies. While lockdown alleviates health crisis, it could induce costly and prolonged country economy recovery. We study effects of these policies on cost of lost lives and cost of economic recession.

Eichenbaum-Rebelo-Trabandt model lacks references to unemployment. To compensate for this shortcoming, we use Gali-Smets-Wouters (2012) linear model which embeds unemployment theory of Gali (2011 a, b). The effects of virus are accounted for by imposing a shock to labor supply. We ran several simulations including user's judgmental assumptions on country output.

These two approaches give economists a multifaceted view on economic impact of this epidemic.

XIII. References

Armstrong John, Black Richard, Laxton Douglas, Rose David. (1998), "A robust method for simulating forward-looking models", Journal of Economic Dynamics and Control, v.22, pp. 489-501.

Anderson-Moore Algorithm, https://www.federalreserve.gov/econres/ama-index.htm

Anderson, G. "A Reliable and Computationally Efficient Algorithm for Imposing the Saddle Point Property in Dynamic Models." Journal of Economic Dynamics and Control, 2010, vol. 34, issue 3, pp. 472-489.

Cristina Arellano, Yan Bai (2020), "Deadly Debt Crises: COVID-19 in Emerging Markets", Federal Reserve Bank of Minneapolis, Staff Report 603, https://www.minneapolisfed.org/research/staff-reports/deadly-debt-crises-covid-19-in-emerging-markets

Eichenbaum M.S., Rebelo S., Trabandt M. (2020a), "Epidemics in the Neoclassical and New Keynesian Models". NBER Working Paper 27430, http://www.nber.org/papers/w27430

Eichenbaum M.S., Rebelo S., Trabandt M. (2020b), "The Macroeconomics of Epidemics", NBER Working Paper No. 26882.

Fernando E. Alvarez, David Argente, Francesco Lippi (2020), "A Simple Planning Problem for Covid-19 Lockdown", NBER Working paper 26981, http://www.nber.org/papers/w26981

Francesca Caselli, Francesco Grigoli, Weicheng Lian, Damiano Sandri (2020), "Protecting Lives and Livelihoods with Early and Tight Lockdowns", IMF Working paper.

Dixit, A., Stiglitz, J. (1977), "Monopolistic Competition and Optimum Product Diversity," American Economic Review 67, pp. 297—308, https://www.aeaweb.org/aer/top20/67.3.297-308.pdf

Galí, Jordi (2011a): "The Return of the Wage Phillips Curve," Journal of the European Economic Association.

Galí, Jordi (2011b): Unemployment Fluctuations and Stabilization Policies: A New Keynesian Perspective, MIT Press (Cambridge, MA).

Gali J., Smets F., and Wouters R. (2012). "Unemployment in an Estimated New Keynesian Model". NBER Working Paper 17084, https://www.nber.org/papers/w17084

Goodman Jonathan and Weare Jonathan (2010), "Ensemble Samplers with Affine Invariance", Communications in Applied Mathematics and Computational Science, Volume 5, Number 1, pp. 65-80, https://projecteuclid.org/download/pdf_1/euclid.camcos/1513731992

Kermack, W., and A. McKendrick. (1927), "A contribution to the mathematical theory of epidemics", Proceedings of the Royal Society of London, series A., pp.115:700–721.

Klein P. (1997). "Using the Generalized Schur Form to Solve a System of Linear Expectational Difference Equations," discussion paper, IIES, Stockholm University, klein@iies.su.se.

Juillard Michel, Laxton Douglas, McAdam Peter, Pioro Hope (1998), "An algorithm competition: First-order iterations versus Newton-based technique", Journal of Economic Dynamics and Control, No. 22, pp.1291—1318, https://www.sciencedirect.com/science/article/pii/S016518899800013X

Michal Andrle (2007). "Linear Approximation to Policy Function in IRIS Toolbox". https://michalandrle.weebly.com/uploads/1/3/9/2/13921270/michal_iris_solve.pdf.

Mihailov A. (2020), "Quantifying the Macroeconomic Effects of the COVID-19 Lockdown: Comparative Simulations of the Estimated Galí-Smets-Wouters Model". Department of Economics, Economic Analysis Research Group (EARG). Discussion Paper No. 2020-07, www.reading.ac.uk

Mihailov, A., Razzu, G., Wang, Z. (2019), "Heterogeneous Effects of Single Monetary Policy on Unemployment Rates in the Largest EMU Economies," Department of Economics, University of Reading, Economics & Management Discussion Paper 2019-07.

Michael Binder and M. Hashem Pesaran. (1997), "Multivariate rational expectations models. Characterization of the Nature of the Solutions and Their Fully Recursive Computation." https://www.jstor.org/stable/pdf/3532897.pdf?refreqid=excelsior%3A5e82c86594ec2406dc1607c3c337e0

Paul R. Miles. (2019), "pymcmcstat: A Python Package for Bayesian Inference Using Delayed Rejection Adaptive Metropolis", The Journal of Open Source Software, https://joss.theoj.org/papers/10.21105/joss.01417

Sebastien Villemot (2011). "Solving Rational Expectations Models at First Order: What Dynare Does." https://www.dynare.org/wp-repo/dynarewp002.pdf

Smets, Frank and Wouters, Rafael. (2007), "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach", American Economic Review, 97(3), pp.586-606.

Coronavirus Source Data, https://ourworldindata.org/coronavirus-source-data

COVID-19 Cases in the United States, https://www.worldometers.info/coronavirus/country/us/#graph-cases-daily

Epidemiological models overview, https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology#The_SIR_model

XIV. APPENDICES

A. Non-Linear System

We are solving a system of equations,

$$f(y) = 0, \text{ where} \tag{A.1}$$

$$f(x_{t-1}, x_t, x_{t+1}, e_t) = \begin{cases} g_1\big(x_{t-1}, x_t, E(x_{t+1}), e_{1,t}\big) \\ \dots \\ g_N\big(x_{t-1}, x_t, E(x_{t+1}), e_{N,t}\big) \end{cases} \quad \text{and} \quad y_t = \begin{cases} x_{t-1} \\ x_t \\ x_{t+1} \end{cases}$$

Here E is the expectation operator. The boundary conditions are:

$$x_0 = starting\ values; x_{T+1} = steady\ state\ values$$

Equations (A.1) are general equations for variables with a maximum lead and lag of one period. If variables lead and lag are larger than one, these equations can be rewritten in the form of (A.1) by introducing new variables. For example, if variable x_{t+2} is presented in (A.1) than one can introduce a new variable y, add a new equation $y_t = x_{t+1}$, and rewrite original equations in the form of (A.1).

We apply an iterative algorithm and linearize equations (A.1). At iteration k:

$$f\big(x_{t-1}^k, x_t^k, E(x_{t+1}^k)\big) + \frac{\partial f^k}{\partial x_{t-1}}\big(x_{t-1}^{k+1} - x_{t-1}^k\big) + \frac{\partial f^k}{\partial x_t}\big(x_t^{k+1} - x_t^k\big) + \frac{\partial f^k}{\partial x_{t+1}}\big(E(x_{t+1}^{k+1}) - E(x_{t+1}^k)\big) = 0 \tag{A.2}$$

These equations are linear with respect to next iteration variables $x_{t-1}^{k+1}, x_t^{k+1}, x_{t+1}^{k+1}$. Equations (A.2) can be rewritten as,

$$L_t \Delta x_{t-1} + C_t \Delta x_t + F_t E(\Delta x_{t+1}) = -f_t \tag{A.3}$$

By stacking Jacobians L_t, C_t, F_t , equations (A.3) can be represented in a matrix form:

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ L_1 & C_1 & F_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_2 & C_2 & F_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_3 & C_3 & F_3 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & L_{T-2} & C_{T-2} & F_{T-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{T-1} & C_{T-1} & F_{T-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L_T & C_T & F_T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} \Delta x_0 \\ \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ . \\ \Delta x_{T-2} \\ \Delta x_{T-1} \\ \Delta x_T \\ \Delta x_{T+1} \end{pmatrix} = -\begin{pmatrix} 0 \\ f_1 \\ f_2 \\ f_3 \\ . \\ f_{T-2} \\ f_{T-1} \\ f_T \\ 0 \end{pmatrix} \tag{A.4}$$

The size of this matrix is $N(T+2)$ by $N(T+2)$. Inverting this matrix could be problematic for large number of equations N, or large time horizon T. This matrix is a sparse matrix. One can use sparse matrices linear algebra scipy package to solve these equations.

Equations (A.4) are solved iteratively until solution converges.

Another approach is an application of LBJ method. It is briefly described below. The starting values of endogenous variables are $\Delta x_0 = 0$. By substituting these values in (A.3) we get:

$\Delta x_1 = M_1 \Delta x_2 + d_1$, where $M_1 = -C_1^{-1} F_1$ and $d_1 = -C_1^{-1} f_1$.

We can find the following expressions by repeating these steps:

$$M_t = -(C_t + L_t M_{t-1})^{-1} F_t$$
$$d_t = -(C_t + L_t M_{t-1})^{-1} (f_t + L_t d_{t-1}) \tag{A.5}$$

Solution can be easily obtained by backward substitution:

$$\Delta x_t = M_t \Delta x_{t+1} + d_{t,} \quad x_{T+1} = steady\ state \tag{A.6}$$

While initial conditions are fixed and $\Delta x_0 = 0$, terminal conditions are set to steady state values. This requires one to find a steady state solution of system (A.1). A reasonable assumption would be to assume that the terminal conditions are floating, and that solution does not change in time, i.e., $\Delta x_{T+1} = \Delta x_T$. We can find formula for the terminal condition by substituting it in (A.6):

$$\Delta x_{T+1} = (I - M_T)^{-1} d_T \tag{A.7}$$

The LBJ algorithm consists of two iterative steps: firstly, compute matrices M and vector d, and secondly, compute solution x. These steps are repeated until numerical solution converges.

Yet another assumption is that endogenous variables at right boundary are fixed. Then the terminal condition is: $\Delta x_{T+1} = 0$.

Below we describe a modified LBJ method. By substituting this expression in (A.3) we get,

$$\Delta x_T = M_T \Delta x_{T-1} + d_T, \text{ where } M_T = -C_T^{-1} L_T \text{ and } d_T = -C_T^{-1} f_T \tag{A.8}$$

Solution can be easily obtained by forward substitution:

$$\Delta x_{t+1} = M_{t+1} \Delta x_t + d_{t+1}\ ;\ \ \Delta x_0 = 0 \tag{A.9}$$

B. Linear System

Equations (A.1) are linear with respect to endogenous and exogenous variables. These equations can be rewritten as,

$$Lx_{t-1} + Cx_t + \mathrm{F}x_{t+1} = -\mathrm{f} - \psi e_t \qquad \text{(B.1)}$$

Binder and Pesaran Method

According to Binder & Pesaran (1997), if the unique and stable solution exists, it is given by:

$$x_t = \mathrm{B}\, x_{t-1} + H\ \mathrm{F} + \sum_{i=0}^{\infty} H^i\, \psi\, E(e_{t+i}) \qquad \text{(B.2)}$$

Where $H = C(C + FB)^{-1}$ and B satisfies quadratic matrix equation:

$$\mathrm{F\ B}^2 + \mathrm{C\ B} + \mathrm{L} = 0 \qquad \text{(B.3)}$$

Authors employ iteration technique to solve this quadratic equation. In general, this matrix equation can have many solutions, a unique solution, or no solution at all.

Anderson-Moore Method

Anderson-More algorithm requires no special treatment for models with endogenous variable with lags and leads greater than one period. This method distinguishes AIM method from all others, where one should introduce new variables and cast model in a form with at most one lead and one lag. For one lead and one lag period endogenous variables, authors provide a simplified solution in the form:

$$x_t - x_0 = \mathrm{B}\,(x_{t-1} - x_0) + \sum_{i=0}^{\infty} F^i \Phi\, \psi\, E(e_{t+i}) \qquad \text{(B.4)}$$

Here B is the reduced form coefficients' matrix, Φ is the exogenous shock scaling matrix, and F is the exogenous shock transfer matrix. According to Anderson, AIM procedure exhibits significant computational performance for large scale models in terms of CPU time.

Jaromír Beneš Method

This method uses QZ matrix factorization. Vector of endogenous variables x_t can be partitioned into predetermined part x_t^P and non-predetermined part x_{t+1}^N:

$$x_t = \begin{cases} x_t^P \\ x_{t+1}^N \end{cases} \qquad \text{(B.5)}$$

And equation (B.1) can be recast in the form,

$$Ax_{t+1} + Bx_t + C + \psi e_t = 0 \qquad \text{(B.6)}$$

Following Klein (1997), the system matrices A, B are decomposed by applying a Generalized Schur algorithm:

$$A = QTZ^H$$
$$B = QSZ^H \tag{B.7}$$

Here Q and Z are the unitary matrices, Z^H is the conjugate transpose of matrix Z, and T, S are upper triangular matrices. Following Michal Andrle (2007), when shocks are not anticipated, the system of equations (B.6) has a solution in the state-space form:

$$\begin{pmatrix} x_t^N \\ \alpha_t \end{pmatrix} = \begin{pmatrix} T^F \\ T^A \end{pmatrix} \alpha_{t-1} + \begin{pmatrix} R^F \\ R^A \end{pmatrix} e_t + \begin{pmatrix} K^F \\ K^A \end{pmatrix}$$
$$x_t^P = \mathrm{U}\, \alpha_t \tag{B.8}$$

The transient matrix T, the shock matrix R and the constant vector K are given below:

$$\begin{aligned}
T^F &= Z_{21} \\
T^A &= -T_{11}^{-1} S_{11} \\
R^F &= (Z_{21}G + Z_{22})\, R^U \\
R^A &= -T_{11}^{-1} [\psi_1 + (S_{11}G + S_{12})\, R^U] \\
G &= -Z_{11}^{-1} Z_{12} \\
R^U &= -S_{11}^{-1} \psi_2 \\
U &= Z_{11} \\
K_u &= (T_{22} + S_{22})^{-1} C_2 \\
X_{a0} &= T_{11}^{-1} (S_{11}G + S_{12}) \\
X_{a1} &= G + T_{11}^{-1} T_{12} \\
K^F &= -(Z_{21}G + Z_{22})\, K_u \\
K^A &= -(X_{a0} + X_{a1})K_u - T_{11}^{-1} C_1
\end{aligned} \tag{B.9}$$

Indices 1 and 2 denote part of matrices T, R and vector K that correspond to predetermined and non-predetermined transition variables. When shocks are anticipated, solution (B.8) is augmented with future shocks:

$$\begin{pmatrix} x_t^N \\ \alpha_t \end{pmatrix} = T\, \alpha_{t-1} + R \begin{pmatrix} e_t \\ e_{t+1} \\ \dots \\ e_{t+N} \end{pmatrix} + \begin{pmatrix} K^F \\ K^A \end{pmatrix} \tag{B.10}$$

Here R is matrix of current and future shocks:

$$R = \begin{bmatrix} R^F & X^F R^U & X^F J\, R^U & X^F J^2\, R^U & \dots & X^F J^{N-1}\, R^U \\ R^A & X^F R^U & X^A J\, R^U & X^A J^2\, R^U & \dots & X^A J^{N-1}\, R^U \end{bmatrix} \tag{B.11}$$

Auxiliary vectors and matrices are shown below:

$$
\begin{aligned}
X^F &= Z_{21}G + Z_{22} \\
X^A &= X_{a1} + J\ X_{a0} \qquad \text{(B.12)} \\
J &= -S_{22}^{-1}\, T_{22}
\end{aligned}
$$

C. Judgmental Adjustments

In many cases user may have her/his view on a path of endogenous variables. We briefly describe the methodology that can be used to forecast variables with anticipation and without anticipation.

Suppose that shocks at times $t, t+1, \dots, t+N$ are anticipated. Then, we can write equations at time $t+1$ as,

$$x_{t+1} = T\, x_t + K + R_0 e_t + R_1 e_{t+1} + R_2 e_{t+2} + \cdots + R_N e_{t+N} \qquad \text{(C.1)}$$

At time $t+2$:

$$x_{t+2} = T\, x_{t+1} + K + R_0 e_{t+1} + R_1 e_{t+2} + R_2 e_{t+3} + \cdots + R_{N-1} e_{t+N}\ , \qquad \text{(C.2)}$$

Or,

$$x_{t+2} = T\,(T\, x_t + K + R_0 e_t + R_1 e_{t+1} + R_2 e_{t+2} + \cdots + R_N e_{t+N}) + K + R_0 e_{t+1} + R_1 e_{t+2} + R_2 e_{t+3} + \dots + R_{N-1} e_{t+N}$$

And,

$$x_{t+2} = T^2 x_t + (T+I)K + TR_0 e_t + (TR_1 + R_0)e_{t+1} + (TR_2 + R_1)e_{t+2} + \cdots + (TR_2 + R_{N-1})e_{t+N}$$

By induction, we can derive that at time $t+k$ the following equation holds:

$$x_{t+k} = T^k x_t + (T^{k-1} + T^{k-2} + \cdots + T^0)K + S_0^k e_t + S_1^k e_{t+1} + S_2^k e_{t+2} + \cdots + S_N^k e_{t+N}$$

Or,

$$x_{t+k} = T^k x_t + (I-T)^{-1}\,(I - T^k)K + \textstyle\sum_{i=0}^{N} S_i^k e_{t+i} \qquad \text{(C.3)}$$

The aggregated shock matrix S is:

$$S_i^k = \begin{cases} T\, S_i^{k-1}, & if\ k > i+1 \\ T\, S_i^{k-1} + R_{i+1-k}, & if\ k \le i+1 \end{cases} \qquad \text{and} \qquad S_i^1 = R_i \qquad \text{(C.4)}$$

One can solve equations (C.3) for values of the future shocks. These shocks will bring path of endogenous variables to the desired level. In other words, the future shocks are “endogenized”, and the corresponding variables are “exogenized”. This numerical procedure can be optimized in terms of CPU speed and memory usage.

Suppose that one has a specific view on the path of endogenous variables, which are given by, $\hat{x}_{t+k}$. Then, writing equation (C.3) for deviations of the endogenous variables from this path, one can find adjustments to the future shocks Δe_{t+k}:

$$x_{t+k} - \hat{x}_{t+k} = S_1^k \Delta e_{t+1} + S_2^k \Delta e_{t+2} + \cdots + S_N^k \Delta e_{t+N} \tag{C.5}$$

By solving equation (C.5), one can find values of a new shock $\hat{e}_{t+k} = e_{t+k} + \Delta e_{t+k}$. This shock brings path of x_{t+k} to the desired level of $\hat{x}_{t+k}$.

Equation (C.5) simplifies when there is only one unexpected shock at time $t + k$:

$$x_{t+k} - \hat{x}_{t+k} = R_0 \, \Delta e_{t+k} \tag{C.6}$$

It can be easily solved. The new value of this shock is:

$$\hat{e}_{t+k} = e_{t+k} + R_0^{-1}(x_{t+k} - \hat{x}_{t+k}) \tag{C.7}$$

If there are several unexpected shocks, then the first shock is adjusted at the first occurrence of unexpected shock and the endogenous variables are computed onwards, then the second shock is adjusted and the forecast is re-evaluated, and so on... This procedure is successively repeated until all unexpected shocks are accounted for.

D. Metrics

Below we show comparison of CPU time and memory footprint of running a small open economy model in Python Platform and in IRIS and DYNARE toolboxes. This model is a DSGE GAP model of Ghana country. It consists of 82 equations describing macroeconomic variables.

The plot below shows data obtained by profiling tools of Anaconda Spyder and Matlab applications. These graphs demonstrate performance benefits of Python Platform when running small and medium sizes DSGE models.

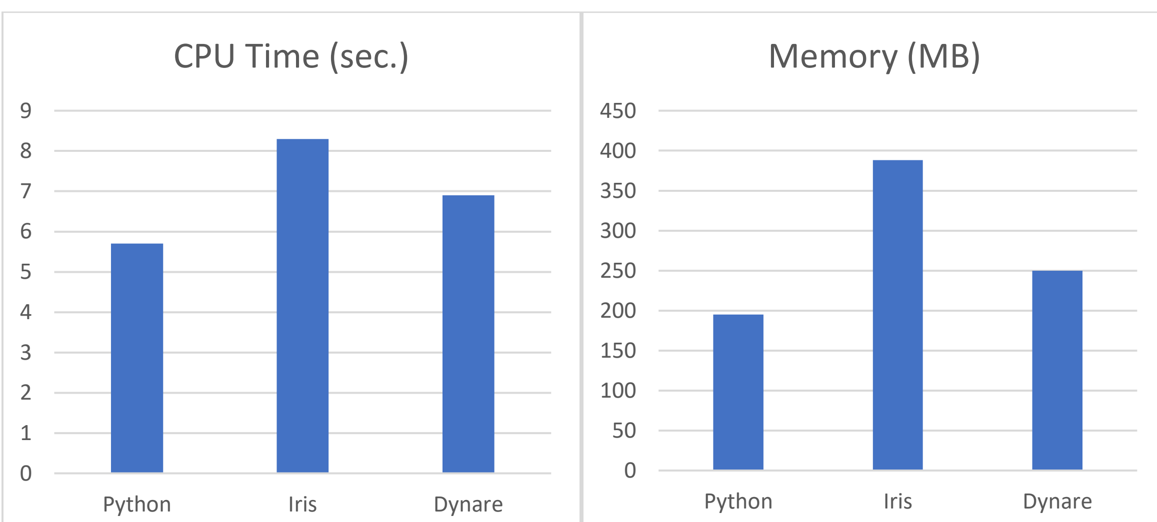

Fig. D1. Total CPU time and memory usage of running Kalman filter in Python Platform, IRIS and DYNARE toolboxes.