Homogeneous

March 31, 2016

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0.1 Table of Contents

0.2 1. Homogeneous Phases of Two Component SO Coupled BECs

0.2.1 1.1 Hamiltonian

To study homogeneous phases, it is conevient to rotate the phase of the two components to transfer the spin-orbit coupling to the kinetic energy. The bare Hamiltonian has the form

$$\mathbf{H} = \begin{pmatrix} \frac{\hbar^2 k^2}{2m} + V_a & \frac{\Omega}{2} e^{2\mathrm{i}k_r x} \\ \frac{\Omega}{2} e^{-2\mathrm{i}k_r x} & \frac{\hbar^2 k^2}{2m} + V_b \end{pmatrix}, \tag{0.1}$$

$$V_a = -\mu - \frac{\delta}{2} + g_{aa} n_a + g_{ab} n_b, \qquad V_b = -\mu + \frac{\delta}{2} + g_{ab} n_a + g_{bb} n_b, \tag{0.2}$$

$$V_+ = -\mu + \frac{(g_{aa} + g_{ab}) n_a + (g_{bb} + g_{ab}) n_b}{2}, \qquad V_- = -\delta + \frac{(g_{aa} - g_{ab}) n_a - (g_{bb} - g_{ab}) n_b}{2}. \tag{0.3}$$

One defines a transformed state:

$$\Psi = e^{\mathrm{i}k_R x \sigma_z} \tilde{\Psi} \tag{0.4}$$

so that $\tilde{\Psi}$ has the effective Hamiltonian

$$\tilde{\mathbf{H}} = \begin{pmatrix} \frac{\hbar^2 (k+k_R)^2}{2m} + V_a & \frac{\Omega}{2} \\ \frac{\Omega}{2} & \frac{\hbar^2 (k-k_R)^2}{2m} + V_b \end{pmatrix}. \tag{0.5}$$

In homogeneous matter, this has eigenstates with energy:

$$\tilde{\Psi} = e^{\mathrm{i}kx} \begin{pmatrix} u_k \\ v_k \end{pmatrix}, \qquad \Psi = \begin{pmatrix} e^{\mathrm{i}(k+k_R)x} u_k \\ e^{\mathrm{i}(k-k_R)x} v_k \end{pmatrix}, \qquad E_k = \hbar^2 \frac{k^2 + k_R^2}{2m} + V_+ - \sqrt{\left(\frac{\hbar^2 k k_r}{m} + V_-\right)^2 + \frac{\Omega^2}{4}}$$

(0.6)

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Noting that V_+ is just an overall shift in energy, we set $V_+ = 0$ (by choosing μ appropriately), use units where $\hbar = m = k_R = 1$ and define $w = \Omega/2$ and $d = V_- \approx -\delta/2$ (see below). (*In the following anaylsis we assume $d > 0$ since this minimizes signs, but in the experiments one typically has $d < 0$*). The dispersion relationship for the lower band is thus characterized by the two parameters w and d which characterize the mixing and detuning respectively:

$$E_k = \frac{k^2 + 1}{2} - \sqrt{(k+d)^2 + w^2}, \qquad \frac{u_k}{v_k} = \frac{k + d - \sqrt{(k+d)^2 + w^2}}{w} = \frac{-w}{k + d + \sqrt{(k+d)^2 + w^2}},$$

(0.7)

$$\frac{n_a}{n_b} = \frac{\sqrt{(k+d)^2 + w^2} - (k+d)}{\sqrt{(k+d)^2 + w^2} + (k+d)} = \frac{1-K}{1+K}, \qquad K = \frac{k+d}{\sqrt{(k+d)^2 + w^2}}$$

(0.8)

Note that the densities are related to the square of the eigenvectors: $n_a = n_0 u_k^2$ and $n_b = n_0 v_k^2$. This relates the polarization of the system to the quasi-momentum k:

$$K = \frac{n_b - n_a}{n_b + n_a} = \frac{k+d}{\sqrt{(k+d)^2 + w^2}}.$$

(0.9)

The total momentum density of the system is thus:

$$p = (k+1)n_a + (k-1)n_b = (n_a + n_b)\frac{(1+k)(1-K) - (1-k)(1+K)}{2}.$$

(0.10)

0.2.2 1.2 Galilean Covariance

Consider a single component system with external potential $V(\vec{x})$. Galilean covariance is expressed as the invariance under the following transformation (boost) when $V(\vec{x}) = V_0$ does not depend on $\vec{x}$:

$$\psi_{\vec{v}}(\vec{x}, t) = e^{-\mathrm{i}\phi}\psi(\vec{x} + \vec{v}t, t), \qquad \hbar\phi = m\vec{v} \cdot \vec{x} + \frac{1}{2}mv^2 t.$$

(0.11)

If you have not seen this before, check explicitly that $\psi_{\vec{v}}$ satisfies:

$$\mathrm{i}\hbar\dot{\psi}_{\vec{v}} = \left(\frac{-\hbar^2\nabla^2}{2m} + V(\vec{x} + \vec{v}t)\right)\psi_{\vec{v}}.$$

(0.12)

The phase is chosen to exactly cancel the extra terms generated by differentiating with the new $\vec{v}t$ dependence in the argument of $\psi_{\vec{v}}$.

In our system, the corresponding transformation gives:

$$\mathbf{H} = \begin{pmatrix} \frac{\hbar^2 k^2}{2m} + V_a & \frac{\Omega}{2}e^{2\mathrm{i}k_r(x+vt)} \\ \frac{\Omega}{2}e^{-2\mathrm{i}k_r(x+vt)} & \frac{\hbar^2 k^2}{2m} + V_b \end{pmatrix}$$

(0.13)

Thus, one can again define a transformed state:

$$\Psi = e^{\mathrm{i}k_R(x+vt)\sigma_z}\tilde{\Psi}. \tag{0.14}$$

The time dependence here will add a term to the Schrödinger equation which will adjust the detuning:

$$\mathrm{i}\hbar\partial_t\Psi = e^{\mathrm{i}k_R(x+vt)\sigma_z}(\mathrm{i}\hbar\partial_t - \hbar k_R v\sigma_z)\tilde{\Psi}. \tag{0.15}$$

so that $\tilde{\Psi}$ has the effective Hamiltonian

$$\tilde{\mathbf{H}} = \begin{pmatrix} \frac{\hbar^2(k+k_R)^2}{2m} + V_a + \hbar k_R v & \frac{\Omega}{2} \\ \frac{\Omega}{2} & \frac{\hbar^2(k-k_R)^2}{2m} + V_b - \hbar k_R v \end{pmatrix}. \tag{0.16}$$

In other words, the detuning depends on the velocity of the local frame:

$$\delta_{\vec{v}} = \delta - 2\hbar k_R v, \qquad d_v = d + 2v \tag{0.17}$$

where v is measured in units of $\hbar k_R/m = 1$.

Note: I always find the signs of v confusing. I think this is correct: if v is positive, it corresponds to a portion of the system moving right so that locally any external potential is seen moving to the left.

0.2.3 1.3 Ground State

The ground state is characterized by the minimum momentum k_0:

$$\frac{\partial E_k}{\partial k} = 0 = k_0 - \frac{k_0 + d}{\sqrt{(k_0 + d)^2 + w^2}} = k_0 - K(k_0), \tag{0.18}$$

$$\frac{u_{k_0}}{v_{k_0}} = \sqrt{\frac{n_a}{n_b}} = \frac{(k_0 + d)(k_0 - 1)}{wk_0} = \frac{-wk_0}{(k_0 + d)(k_0 + 1)}, \qquad \frac{n_a}{n_b} = \frac{1 - k_0}{1 + k_0}. \tag{0.19}$$

Since $K = k_0$ at the minimum, the total momentum density of the ground state is zero as expected.

We can also invert these relationships and express:

$$k_0 = \frac{n_b - n_a}{n_b + n_a}, \qquad w^2 = (k_0^{-2} - 1)(k_0 + v)^2, \tag{0.20}$$

$$w = \frac{m\Omega}{\hbar^2 k_R^2}, \qquad v = m\frac{-\delta + (g_{aa} - g_{ab})n_a - (g_{bb} - g_{ab})n_b}{\hbar^2 k_R^2} \approx \frac{-m\delta}{\hbar^2 k_R^2}. \tag{0.21}$$

This tells us what the minimum quasi-momentum k_0 and mixing w are for a given density ratio and detuning v. Note that for Rubidium, $g_{ab} \approx g_{aa} \approx g_{bb}$, so it is a very good approximation to drop the non-linear terms in the previous equations:

$$d \approx \frac{-m\delta}{\hbar^2 k_R^2}. \tag{0.22}$$

This means that the density fraction and minimum quasi-momentum k_0 are approximately constant throughout the entire system, even if the density is changing because of a slowly varying external potential.

The overall density scale is then determined by setting $E_k = 0$ by adjusting μ as usual. (Note: that in the equation below, μ also includes an offset from v_+:

$$\frac{n_b}{n_a} = \frac{1 + k_0}{1 - k_0}, \qquad n_a = \frac{2(\mu - V_{\text{ext}})}{g_{aa} + g_{ab} + (g_{bb} + g_{ab})\frac{n_b}{n_a}} \tag{0.23}$$

0.3 2. Single Branch Physics

We now assume that only the lowest branch is occupied. This is true for the ground state of the system, but also for physical systems where one does not introduce too much energy.

0.3.1 2.1 Probability Current

Consider the density matrix $\mathbf{R} = |\psi\rangle \langle\psi|$ such that the local density is $n(x) = \langle x|\mathbf{R}|x\rangle$. The time derivatives are given by the usual commutation relations following from the GPE:

$$\dot{\mathbf{R}} = \frac{i}{\hbar}[\mathbf{R}, \mathbf{H}], \qquad \dot{n}(x) = \langle x|\dot{\mathbf{R}}|x\rangle = -\nabla \cdot j, \tag{0.24}$$

where $j = n(x)v$ is the probability current:

$$j(x) = \langle x|\mathbf{j}|x\rangle. \tag{0.25}$$

One must be careful about defining $\mathbf{j}$. With the usual quadratic dispersion, one has $\mathbf{j} = \frac{1}{2m}\{\mathbf{p}, \mathbf{R}\} = \frac{1}{2}\{\omega'(\mathbf{p}), \mathbf{R}\}$ where $\mathbf{p}$ is the momentum operator and $\omega(\mathbf{p}) = \mathbf{p}^2/2m$ is the kinetic energy (single-particle dispersion relation), but this is not valid with general dispersion. Instead, one must generalize the anticommutator into an averaging over orders so that:

$$S(\mathbf{p}^n\mathbf{R}) = \frac{\mathbf{p}^n\mathbf{R} + \mathbf{p}^{n-1}\mathbf{R}\mathbf{p} + \cdots \mathbf{p}\mathbf{R}\mathbf{p}^{n-1} + \mathbf{R}\mathbf{p}^n}{n + 1}, \qquad \mathbf{j} = S(\omega'(\mathbf{p})\mathbf{R}). \tag{0.26}$$

I do not yet know of a good/efficient way of computing the symmetric-ordered product, so in the code I compute the velocity by integrating numerically the time-derivative of the density which can easily be computed by applying the Hamiltonian:

$$j(x) = -\int \mathrm{d}x \dot{n}(x). \tag{0.27}$$

This becomes more complicated in higher dimensions but works very well in 1D.

(I could not find much in the literature about the required symmetric ordering, but am sure this must be discussed many places. There is an example in Drouhin:2009 but it does not make much sense.)

For homogeneous systems, the density operator will commute with the momentum operator, we do not need to consider the complications with symmetric ordering and can simply write the physical momentum:

$$p = \frac{\mathrm{d}E_k}{\mathrm{d}k} = k - \frac{k + d}{\sqrt{(k + d)^2 + w^2}} = k - K(k) = k - \frac{n_b - n_a}{n_b + n_a}. \tag{0.28}$$

In addition to the two-component system, we simulate a single-component system with modified dispersion relationship

$$\omega(k) = E_k(k - k_0). \qquad (0.29)$$

One can directly compare this to the two-component simulation using $n_{a,b} = n(1 \mp K)/2$ to compute the population fractions where $K = K(k - k_0) = (n_b - n_a)/(n_b + n_a)$.

0.3.2 2.2 Hydrodynamics

For the hydrodynamic formulation of the problem we start by effecting a Madelung transformation: $\psi = \sqrt{n}e^{i\phi}$. Collecting real and imaginary parts of the NLSEQ we obtain:

$$\dot{n} = \frac{2}{\hbar}\Im\psi^\dagger\omega(\mathbf{p})\psi = -\nabla \cdot j, \qquad (0.30)$$

$$\hbar\dot{\phi}n = -\Re\psi^\dagger\omega(\mathbf{p})\psi - V(n)n \qquad (0.31)$$

from the probability current discussed above:

$$\dot{n} = -\nabla j, \qquad j(x) = \langle x|\mathbf{j}|x\rangle = \langle x|S(\omega'(\mathbf{p})\mathbf{R})|x\rangle \qquad (0.32)$$

For now we simplfy to one dimension so $\nabla\phi = \phi'$. Then we have the following explicit expressions with $k = \hbar\phi'$ as the quasi-momentum and quasi-velocity $u = k/m$ where m is the mass of the particles (not the effective mass):

$$v = \frac{j}{n} = \omega'(k) - A[k, n] \qquad \text{(group velocity)}$$

$$\hbar\dot{\phi} = -\omega(k) - B[k, n] - V(n). \qquad (0.33)$$

where A and B depend on k and n and various derivatives. Importantly, both $A = B = 0$ for homogeneous states. (Expressions for the quartic dispersion are given below). We thus have a hydrodynamic description in the following from:

$$\dot{\rho} = m\dot{n} = -\nabla(nk), \qquad (0.34)$$

$$v + A = \omega'(k) \qquad (0.35)$$

$$\frac{\partial(v + A)}{\partial t} + (v + A) \cdot \nabla(v + A) = \omega''(k)\nabla[-V(n) - B]. \qquad (0.36)$$

Here we see the role of the inverse effective mass $m^* = 1/\omega''(k)$. This kind of looks like a standard hydrodynamic description with effective velocity of the fluid is $v + A$. In the last equation we see that B acts as an additional potential, and it includes the usual quantum pressure term along with higher order corrections. Note, however, that we can't use this as an actual velocity because then the conservation equation will be violated.

Here are the forms of A and B for a quadratic dispersion $\omega(\mathbf{p}) = a\mathbf{p} + b\mathbf{p}^2 + c\mathbf{p}^3 + d\mathbf{p}^4$. Note that A has dimensions of velocity and B dimensions of energy. Note also that both A and B are proportional to $\hbar^2$ indicating that they are purely quantum effects. (This should be apparent since they arise from the required symmetric ordering.) Note also that the expressions do not depend on the scaling of n: I think they should probably be expressed in terms of the mass density $\rho = mn$ so that the hydrodynamic equations do not explicitly involve m, relying instead on the effective mass m^*.

$$\Re\psi^\dagger\omega(\mathbf{p})\psi = n\left(\omega(k) - B\right) \tag{0.37}$$

$$\frac{A[k,n]}{\hbar^2} = \left(\frac{n''}{n} - \frac{3n'^2}{4n^2}\right)\frac{w'''(k)}{6} + 2d\frac{(nk')'}{n} \tag{0.38}$$

$$= \frac{1}{n^{1/4}}\left(\frac{\sqrt{n}'}{n^{1/4}}\right)'\frac{w'''(k)}{3} + 2d\frac{(nk')'}{n} \tag{0.39}$$

$$\frac{B[k,n]}{\hbar^2} = \frac{\sqrt{n}''}{\sqrt{n}}\frac{\omega''(k)}{2} + \left(k'' + \frac{3\sqrt{n}'k'}{\sqrt{n}}\right)\frac{\omega'''(k)}{6} - d\left(3k'^2 - \frac{\sqrt{n}''''}{\sqrt{n}}\right) \tag{0.40}$$

$$= b\left(\frac{\sqrt{n}''}{\sqrt{n}}\right) + c\left(k'' + \frac{3(\sqrt{n}'k)'}{\sqrt{n}}\right) + d\left(4kk'' + 3k'^2 + \frac{6(\sqrt{n}'k^2)'}{\sqrt{n}} - \frac{\sqrt{n}''''}{\sqrt{n}}\right) \tag{0.41}$$

The first term in B is the usual "Quantum Pressure" term. Note that it also has a factor of the inverse effective mass.

2.2.1 Additional Equations

I tried to massage these a bit into the form (1.3) of El and Hoefer, but the dependencies are not correct. (In the following two lines, factors of $\hbar = m = 1$ are missing.)

$$\tilde{v} = v + A \tag{0.42}$$

$$n_{,t} + (n\tilde{v})_{,x} = (-nA)_{,x}, \tag{0.43}$$

$$(n\tilde{v})_{,t} + (n\tilde{v}^2)_{,x} = n\omega''(u)\nabla[-V(n) - B] - \tilde{v}(nA)_{,x} \tag{0.44}$$

$$v + A = \omega'(u) \tag{0.45}$$

$$\tilde{v}_{,t} + \frac{1}{2}(\tilde{v}^2)_{,x} = \omega''(u)\nabla[-V(n) - B], \tag{0.46}$$

$$(n\tilde{v})_{,t} + (n\tilde{v}^2)_{,x} = \omega''(u)\nabla[-V(n) - B], \tag{0.47}$$

$$\tag{0.48}$$

$$\nabla(v + A) = \omega''(u)\nabla u \tag{0.49}$$

$$\frac{\partial v + A}{\partial t} = \omega''(u)\dot{u} \tag{0.50}$$

$$A = \frac{1}{n^{1/4}}\left(\frac{\sqrt{n}'}{n^{1/4}}\right)'\frac{w'''(u)}{3} + 2d\frac{(nu')'}{n} + \cdots, \tag{0.51}$$

$$B = \frac{\sqrt{n}''}{\sqrt{n}}\frac{\omega''(u)}{2} + \left(u'' + \frac{3\sqrt{n}'u'}{\sqrt{n}}\right)\frac{\omega'''(u)}{6} + d\left(3u'^2 - \frac{\sqrt{n}''''}{\sqrt{n}}\right) + \cdots \tag{0.52}$$

and the chain

$$\dot{u} = \nabla\dot{\phi} = \nabla\left[-n^{-1}\Re\psi^\dagger\omega(\mathbf{p})\psi - V(n)\right] = \nabla\left[-\omega(u) - B - V(n)\right] = \nabla\left[-V(n) - B\right] - \omega'(u)\nabla u \tag{0.53}$$

$$= \nabla\left[-V(n) - B\right] - \frac{(v + A)\nabla(v + A)}{\omega''(u)} \tag{0.54}$$

0.3.3 2.3 Numerics

Here are the polynomials describing the extrema and inflection points of the dispersion relationships. By squaring these to get rid of the radicals, we end up finding properties for both branches (hence there are 4 extrema, not 3).

$$\frac{\partial E_k}{\partial k} = 0 = k_0 - \frac{k_0 + d}{\sqrt{(k_0 + d)^2 + w^2}} = k_0 - K(k_0), \qquad k_0^2((k_0 + d)^2 + w^2) - (k_0 + d)^2 = 0,$$
(0.55)

$$\frac{\partial^2 E_k}{\partial k^2} = 0 = 1 + \frac{(k_0 + d)^2 - (k_0 + d)^2 - w^2}{\sqrt{(k_0 + d)^2 + w^2}^3} = 1 - \frac{w^2}{\sqrt{(k_0 + d)^2 + w^2}^3}, \qquad k_{\text{inflect}} = \pm\sqrt{w^{4/3} - w^2} - d$$
(0.56)

Here we present some simple code to find the optimal k_0 by using Newton's method to solve the following equation:

$$f(k) = k\sqrt{(k + d)^2 + w^2} - (k + d).$$
(0.57)

This has a nice form that is amenable to solution by Newton's method in a few steps. We start with ansatz $k_0 = \pm 1$.

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Populating the interactive namespace from numpy and matplotlib
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-0.890331418502
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Here is the form of the dispersion relationship:

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/data/apps/anaconda/envs/work/lib/python2.7/site-packages/numpy/core/numeric.py:474: Complex
  return array(a, dtype, copy=False, order=order)
```

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<matplotlib.legend.Legend at 0x10f8a83d0>
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0.4 References

The phase diragram for homogeneous states is discussed in the following papers:

- Ho:2011: For $\delta = 0$, they discuss the emergence of striped phases for different values of the coupling constants. They use parameters $\alpha = g_{ab}/g$, $\beta = (g_{bb} - g_{aa})/g$ where $g = (g_{aa} + g_{bb})/2$.

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- Martone:2012 and Li:2013: At $\delta = 0$: This is a fairly complete discussion of the response of the system (BdG).

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