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Writeup - `siltsong`

1. INTRODUCTION

`siltsong` is a code that takes in a scattering and absorbing medium and renders what it looks like against the mirrors of a telescope. It succeeds `Project Bipolar`, introducing Monte Carlo multiple scattering, physical units instead of intensity ratios, medium emissions, and many algorithmic optimizations. Among the optimizations is an improved peel-off method: at each scattering event, instead of peeling off a portion of the photon's energy and directing it to the observer in situ, the simulation "deposits" the event's pixel location and specific intensity, then performs a total to-observer radiative transfer at the end. As radiative transfer along the sight line of each pixel is only performed once[1], `siltsong` is significantly faster than `Project Bipolar`.

2. COORDINATE SYSTEMS AND THE SCATTERING MEDIUM

`siltsong` inherits the coordinate systems of `Project Bipolar`:

Coordinate System	Description
(px: pixels, py: pixels, d: dw)	A left-handed three-dimensional coordinate system in image space with two axes for pixel location and one axis for depth. The center of the image is aligned to the central source(s). dw= *view_length*/*depth* (see Table 2).
(u: cm, v: cm, w: cm)	The observer's Cartesian coordinates grounded in real space, with the origin positioned at the central source(s) of the object. The u and v axes are aligned parallel to px and py defined in (px, py, d), while the w axis points from the central source(s) to the observer.
(x: cm, y: cm, z: cm)	The object's Cartesian coordinates in real space. The origin is positioned at the central source(s), the z-axis is in the direction of the object's axis of symmetry (if applicable), and the y-axis is parallel to v and py respectively defined in (u, v, w) and (px, py, d).
(r: cm, θ: radians, φ: radians)	The spherical coordinate system also originating at the central source(s). The z-axis is in the direction of the object's axis of symmetry, θ is the angle from the z-axis, and φ is the angle from the x defined in (x, y, z).

Table 1. Descriptions of `siltsong`'s coordinate systems.

The scattering medium can be defined in any coordinate system, and coordinate transformations can be imported from the main `siltsong` package. To run the simulation, the medium must be transformed into both (x, y, z) and (r, θ, φ) before being passed to the `siltsong.radiative_transfer` function.

3. RADIATIVE TRANSFER

Table 2 lists all the input parameters accepted by `siltsong.radiative_transfer`. While the units in the table are expressed in the centimeter-gram-second (cgs) system, the `siltsong.radiative_transfer` function operates correctly with any dimensionally consistent units (e.g., *density_spherical* may be provided in units such as number of stars per

[1] In `Project Bipolar`, we calculate the optical depth for each sight line only once and store them in arrays; `siltsong` saves the array reading time.

cubic parsec). Importable functions such as `siltsong.stars.blackbody` return outputs in cgs units by default. A particularly convenient feature of `siltsong` is its ability to accept multiple components for the scattering medium. The six parameters that specify the medium in Table 2 (between rows *density_spherical* and *scattering_phase_function*) can be passed as lists, allowing the specification of multiple components of the object's medium with distinct properties.

Table 2. Parameters of `siltsong.radiative_transfer`.

Parameter	**Description**
view_length	The physical distance represented by the side length of the image in cm.
inclination_degrees	the angle between the object's axis of symmetry and the line of sight in degrees.
resolution	the number of pixels along one side of the image.
central_source	the source function of the central source(s) in units of erg s^{-1} sr^{-1} cm^{-2} cm^{-1}.
density_spherical	a function or a list of functions specifying the density of the medium with parameters r and θ. In the default convention, the function outputs in units of g cm^{-3}.
density_cartesian	a function or a list of functions specifying the density of the medium with parameters x, y, and z. In the default convention, the function outputs in units of g cm^{-3}. *density_cartesian* should describe the identical medium or medium component as the corresponding *density_spherical*. Coordinate transformations can be imported from the main `siltsong` package.
sca_cm_squared_per_g	a value or a list of values representing the scattering cross sectional area per quantity of scattering medium. In the default convention accepted in units of cm^2 g^{-1}. `siltsong.dust` contains functions that calculate this value based on a given distribution (e.g. the MRN grain size distribution).
ext_cm_squared_per_g	a value or a list of values representing the extinction cross sectional area per quantity of scattering medium. In the default convention accepted in units of cm^2 g^{-1}.
source_function	a value or a list of values representing the source function of the medium. In the default convention accepted in units of erg s^{-1} sr^{-1} cm^{-2} cm^{-1}.
scattering_phase_function	a scattering phase function or a list of scattering phase functions, with input parameter the angle in radians. The Henyey-Greenstein scattering phase function can be imported from `siltsong.dust`.
depth	parameter specifying the number of grid units used by the simulation in the direction of d.
depth_substeps	parameter specifying the number of steps used when integrating within a grid unit in the direction of d. Radiative transfer during multiple scattering uses the same step size.
distance_steps	parameter specifying the number of grid divisions the simulation uses to span a range of *view_length*/2 in the r direction.
distance_substeps	parameter specifying the number of steps used when integrating within a grid unit in the direction of r.
theta_steps	parameter specifying the number of grid divisions the simulation uses to span an angular range of $\pi/2$ in the θ direction.
phi_steps	parameter specifying the number of grid divisions the simulation uses to span an angular range of π in the φ direction.

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Table 2 – *continued from previous page*

Parameter	Description
ms_count	parameter specifying the number of randomly sampled multiple scattering photons.
axisymmetry	Default as "False"; if set as "True", `siltsong` will enable optimizations assuming the medium is axisymmetric with respect to the z axis.
reflection_symmetry	Default as "False"; if set as "True", `siltsong` will enable optimizations assuming the medium is symmetric with respect to the (x, y) plane.
include_central_source_self	Default as "True"; if set as "False", the output image will not contain direct emission from the central source(s).

3.1. *Non-scattered central-originated photons*

`siltsong` considers emissions from both the medium and the central source(s). The specific intensity of the central source(s) is passed to `siltsong.radiative_transfer` as the *central_source* parameter. The simulation will first deposit an emission with specific intensity *central_source* at $\vec{p}_0 = (px = \mathrm{int}(resolution - 1)/2,\, py = \mathrm{int}(resolution - 1)/2,\, d = \mathrm{int}(depth - 1)/2)$ until peel-off is performed[2]:

$$\Delta I_{\nu,\,\mathrm{dep}}(px,\, py,\, d) = central\_source \cdot \delta^3((px,\, py,\, d) - \vec{p}_0)\ . \tag{1}$$

3.2. *Singly-scattered central-originated photons*

For singly-scattered central-originated photons, we begin by considering directions with θ ranging from $\pi/(2 \cdot theta\_steps)$ to $\pi/2$ sampled in uniform steps of $\Delta\theta = \pi/(2 \cdot theta\_steps)$. Given the axisymmetry and reflection symmetry of the object, we don't sample in φ or where θ greater than $\pi/2$ here. With the initial specific intensity equal to the *central_source* parameter, we perform radiative transfer using Eq. 2 along each of the sampled directions. This yields the central-originated (central source(s) + medium) specific intensity at locations within the medium sampled in uniform steps of r from $view\_length/(2 \cdot distance\_steps)$ to $view\_length/2$ with step size $\Delta r = view\_length/(2 \cdot distance\_steps)$. For this radiative transfer calculation we use $S_\nu = source\_function$, $\kappa_\nu = ext\_cm\_squared\_per\_g$ multiplied by the density, and $ds = \Delta r / distance\_substeps$. $\kappa_{\nu,\,\text{non-scattering}}$ is density multiplied by $(ext\_cm\_squared\_per\_g - sca\_cm\_squared\_per\_g)$ since emission from scattered light is handled separately.

$$dI_\nu = -I_\nu \kappa_\nu\, ds + S_\nu \kappa_{\nu,\,\text{non-scattering}}\, ds\ . \tag{2}$$

We now deposit the contributions of singly-scattered central-originated photons. For this step we have to consider the φ direction since the object's axis of symmetry is tilted with respect to the line of sight. We sample in φ with from $\pi/(2 \cdot phi\_steps)$ to $\pi - \pi/(2 \cdot phi\_steps)$ with step size $\Delta\varphi = \pi/phi\_steps$. Combined with the previously sampled θ, the differential specific flux intercepted by this area element is equal to the incoming specific intensity $I_{\nu,\,\mathrm{inc}}$ multiplied by the differential solid angle:

$$dF_\nu(r, \theta, \varphi) = I_{\nu,\mathrm{inc}} \frac{(r \sin\theta\, d\varphi)(r\, d\theta)}{r^2} = I_{\nu,\mathrm{inc}} \sin\theta\, d\theta\, d\varphi\ . \tag{3}$$

Intuitively, the $\sin\theta$ factor compensates our non-isotropic sampling of directions: with uniform sampling in both θ and φ more photons are sampled near smaller θ. With the differential specific flux, the differential scattering optical depth between r to $r + dr$ is given by $\rho\kappa_{\nu,\,\mathrm{sca}}\, dr$, where ρ is the *density_spherical* function evaluated at the current location, and $\kappa_{\nu,\,\mathrm{sca}} = sca\_cm\_squared\_per\_g$. Therefore, the fraction of scattered flux along a path of length ∂r is $(1 - e^{-\rho\kappa_{\nu,\,\mathrm{sca}}\, dr})$. The scattered specific flux contributed by the volume subtending $([r,\, r + dr],\, [\theta,\, \theta + d\theta],\, [\varphi,\, \varphi + d\varphi])$ is thus given by:

$$\frac{dF_{\nu,\mathrm{sca}}}{dr\, d\theta\, d\varphi}(r, \theta, \varphi) = I_{\nu,\mathrm{inc}} \sin\theta\, \frac{1 - e^{-\rho(r,\theta,\varphi)\,\kappa_{\nu,\mathrm{sca}}\, dr}}{dr}\ . \tag{4}$$

[2] For computational convenience the actual deposit is done later in the code.

The differential scattered specific intensity arises from the scattered flux multiplied by the scattering phase function. The scattering phase function is denoted as $p(\gamma)$ in Eq. 5 with γ being the scattering angle toward the observer.

$$\frac{dI_{\nu,\mathrm{sca}}}{dr\,d\theta\,d\varphi}(r,\theta,\varphi) = I_{\nu,\mathrm{inc}}\,\sin\theta\cdot p(\gamma)\cdot\frac{1-e^{-\rho(r,\theta,\varphi)\,\kappa_{\nu,\mathrm{sca}}\,dr}}{dr}\;. \tag{5}$$

If we integrate this expression with respect to r, θ, and φ, we obtain the total in situ contributions of all the central-originated single-scattering trajectories:

$$\begin{aligned} I_{\nu,\text{ central-originated single-scattering}} &= \iiint dI_{\nu,\mathrm{sca}}(r,\theta,\varphi) \\ &\simeq \sum_{i=1}^{N_r}\sum_{j=1}^{N_\theta}\sum_{k=1}^{N_\varphi}\frac{dI_{\nu,\,\mathrm{sca}}}{dr\,d\theta\,d\varphi}(r_i,\theta_j,\varphi_k)\,\Delta r\,\Delta\theta\,\Delta\varphi \\ &= \sum_{i=1}^{N_r}\sum_{j=1}^{N_\theta}\sum_{k=1}^{N_\varphi} I_{\nu,\,\mathrm{inc}}(r_i,\theta_j,\varphi_k)\cdot\sin\theta_j\cdot p(\gamma_{ijk})\cdot\frac{1-e^{-\rho(r,\theta,\varphi)\,\kappa_{\nu,\mathrm{sca}}\,dr}}{dr}\,\Delta r\,\Delta\theta\,\Delta\varphi \\ &\simeq \sum_{i=1}^{N_r}\sum_{j=1}^{N_\theta}\sum_{k=1}^{N_\varphi} I_{\nu,\,\mathrm{inc}}(r_i,\theta_j,\varphi_k)\cdot\sin\theta_j\cdot p(\gamma_{ijk})\cdot(1-e^{-\rho(r_i,\theta_j,\varphi_k)\kappa_{\nu,\,\mathrm{sca}}\,\Delta r})\,\Delta\theta\,\Delta\varphi\;, \end{aligned} \tag{6}$$

where N_r, N_θ, and N_φ are respectively *distance_steps*, *theta_steps*, and *phi_steps*. Thus, for each of the (r, θ, φ) locations we simulate, we find the corresponding [px, $px+1$ pixel], [py, $py+1$ pixel], and [d, $d+\mathrm{dw}$] region, and deposit at (px, py, d) an emission of

$$\begin{aligned} \Delta I_{\nu,\,\mathrm{dep}}(px,py,d) &= \iiint_{\mathcal{R}_{px,py,d}}\frac{dI_{\nu,\,\mathrm{sca}}}{dr\,d\theta\,d\varphi}(r,\theta,\varphi)\,dr\,d\theta\,d\varphi \\ &\simeq \sum_{(r,\theta,\varphi)\in\mathcal{R}_{px,py,d}}\frac{dI_{\nu,\,\mathrm{sca}}}{dr\,d\theta\,d\varphi}(r,\theta,\varphi)\,\Delta r\,\Delta\theta\,\Delta\varphi \\ &\simeq \sum_{(r,\theta,\varphi)\in\mathcal{R}_{px,py,d}} I_{\nu,\,\mathrm{inc}}(r,\theta,\varphi)\cdot\sin\theta\cdot p(\gamma)\cdot\left(1-e^{-\rho(r,\theta,\varphi)\,\kappa_{\nu,\,\mathrm{sca}}\,\Delta r}\right)\,\Delta\theta\,\Delta\varphi\;. \end{aligned} \tag{7}$$

For this expression in the simulation, the multiplicative factors are applied at separate locations to optimize computation time. The result after all deposits are completed is an array representing the specific intensity of emissions from first-scatterings, indexed by their pixel coordinates and depth.

3.3. *Medium-emitted photons and multiple scattering*

By now we have considered the contributions of non-scattered and singly-scattered photons with trajectories originated from the central source(s). We have not accounted for multiple scattering trajectories originated from the central source(s), as well as any photons with trajectories not originated from the central source(s). These photons are computationally evaluated together in `siltsong`; they both involve randomly sampling locations and directions using the Monte Carlo method, and performing radiative transfer using Eq. 2 along those trajectories.

Multiple scattering is performed in the (px, py, d) space: we denote the scattering points here as ($px_{i,j}$, $py_{i,j}$, $d_{i,j}$), where i is the ith sampled photon from 1 to *ms_count* and j is the jth scattering from 1 to ∞. In the simulation, we first sample *ms_count* points; for each of the sampled locations ($px_{i,1}$, $py_{i,1}$, $d_{i,1}$), we sample a random isotropic direction $\Omega_{i,1}$. We in turn sample a distance $r_{i,1}$ to the succeeding scattering event uniformly between 0 and *view_length*. This randomly sampled set of location, direction, and distance, acts both as a trajectory for the second (or third, fourth, ...) scattering of central-originated photons, and a trajectory for the first (second, third, ...) scattering of medium-emitted photons at the sampled location. Similar to Eq. 7, for the central-originated photons, we apply the scattering phase function to obtain the differential scattered specific intensity toward Ω from [$px_{i,1}$, $px_{i,1}+1$ pixel], [$py_{i,1}$, $py_{i,1}+1$ pixel], and [$d_{i,1}$, $d_{i,1}+\mathrm{dw}$]:

$$\frac{dI_{\nu,\text{ after preceding scattering}}}{dpx\,dpy\,dd}(px_{i,1},\,py_{i,1},\,d_{i,1}) \simeq \frac{\sum_{r,\,\theta,\,\varphi\,\in\,\mathcal{R}_{px_{i,1},py_{i,1},d_{i,1}}} I_{\nu,\text{ inc}} \sin\theta \cdot p(\alpha_1)\cdot(1-e^{-\rho\kappa_{\nu,\text{ sca}}\Delta r})\,\Delta\theta\,\Delta\varphi}{1\text{ pixel}\cdot 1\text{ pixel}\cdot \mathrm{dw}}\;, \tag{8}$$

where α_1 is the scattering angle. From here we numerically propagate the photons to the succeeding scattering event at $(px_{i,2},\, py_{i,2},\, d_{i,2})$ using Eq. 2 with $ds = \mathit{view\_length}/(\mathit{depth}\cdot \mathit{depth\_substeps})$. This yields $\frac{dI_{\nu,\text{ before succeeding scattering}}}{dpx\,dpy\,dd}$. At the succeeding scattering event, similarly to Eq. 5, we obtain the scattered specific intensity:

$$\frac{dI_{\nu,\text{ sca}}}{dpx\,dpy\,dd\,d\Omega\,dr}(px_{i,1},\,py_{i,1},\,d_{i,1},\,\Omega_{i,1},\,r_{i,1}) = \frac{dI_{\nu,\text{ before succeeding scattering}}}{dpx\,dpy\,dd}\cdot\frac{(1-e^{-\rho_{i,2}\kappa_{\nu,\text{ sca}}\,dr})}{dr}\cdot p(\gamma_{i,1})\;, \tag{9}$$

with $\rho_{i,2}$ being the density evaluated at $(px_{i,2},\, py_{i,2},\, d_{i,2})$, and $\gamma_{i,1}$ being the scattering angle for the 2nd scattering event. Now, we estimate the total contributions for all photons that first scattered at $(px_{i,j},\, py_{i,j},\, d_{i,j})$ then scattered somewhere else by integrating Eq. 9:

$$\begin{aligned}
\frac{dI_{\nu,\text{ sca}}}{dpx\,dpy\,dd}(px_{i,1},\,py_{i,1},\,d_{i,1}) &= \iint \frac{dI_{\nu,\text{ sca}}}{dpx\,dpy\,dd\,d\Omega\,dr}(px_{i,1},\,py_{i,1},\,d_{i,1},\,\Omega,\,r)\,d\Omega\,dr \\
&= \iint \frac{dI_{\nu,\text{ before succeeding scattering}}}{dpx\,dpy\,dd}\cdot\frac{(1-e^{-\rho\kappa_{\nu,\text{ sca}}\,dr})}{dr}\cdot p(\gamma)\,d\Omega\,dr \\
&= \frac{dI_{\nu,\text{ before succeeding scattering}}}{dpx\,dpy\,dd}\int_0^{4\pi}\int_0^{\infty} p(\gamma)\frac{(1-e^{-\rho(px_{i,1},\,py_{i,1},\,d_{i,1},\Omega,r)\kappa_{\nu,\text{ sca}}\,dr})}{dr}\,d\Omega\,dr \\
&\simeq \frac{dI_{\nu,\text{ before succeeding scattering}}}{dpx\,dpy\,dd}\left(p(\gamma_{i,1})\int_0^{4\pi}d\Omega\right)\left(\frac{(1-e^{-\rho_{i,2}\kappa_{\nu,\text{ sca}}\,\Delta r})}{\Delta r}\int_0^{l}dr\right) \\
&\simeq \frac{dI_{\nu,\text{ before succeeding scattering}}}{dpx\,dpy\,dd}\,4\pi l\cdot p(\gamma_{i,1})\cdot\frac{(1-e^{-\rho_{i,2}\kappa_{\nu,\text{ sca}}\,\Delta r})}{\Delta r}\;,
\end{aligned} \tag{10}$$

using $\Delta r = \mathit{view\_length}/(\mathit{depth}\cdot\mathit{depth\_substeps})$ and $l = \mathit{view\_length}/2$ to estimate the integral[3]. Now, the full in situ contributions from all the medium-originated trajectories and/or multiple scattering trajectories at all possible locations is given by:

$$\begin{aligned}
I_{\nu,\text{ medium-originated and/or multiple scattering}} &= \iiint \sum_{\mathrm{j^{th}\ scattering}}^{\infty}\frac{dI_{\nu,\text{ sca}}}{dpx\,dpy\,dd}(px_{i,j},\,py_{i,j},\,d_{i,j})\,dpx\,dpy\,dd \\
&\simeq \sum_{i=1}^{N_{ms}}\frac{\Delta px\,\Delta py\,\Delta d}{N_{ms}}\sum_{j}^{\infty}\frac{dI_{\nu,\text{ sca}}}{dpx\,dpy\,dd}(px_{i,j},\,py_{i,j},\,d_{i,j})\;,
\end{aligned} \tag{11}$$

with $N_{ms} = \mathit{ms\_count}$; $\Delta px = \mathit{resolution}$, $\Delta py = \mathit{resolution}$, and $\Delta d = \mathit{depth}$ have units pixels, pixels, and dw respectively. Therefore, for each sampled set of location $(px_{i,j},\, py_{i,j},\, d_{i,j})$, direction $\Omega_{i,j}$, and distance $r_{i,j}$, we deposit at the location $(px_{i,j+1},\, py_{i,j+1},\, d_{i,j+1})$ an emission with

$$\Delta I_{\nu,\text{ dep}} = \frac{\Delta px\,\Delta py\,\Delta d}{N_{ms}}\frac{dI_{\nu,\text{ before succeeding scattering}}}{dpx\,dpy\,dd}\,4\pi l\cdot p(\gamma_{i,1})\cdot\frac{(1-e^{-\rho_{i,2}\kappa_{\nu,\text{ sca}}\,\Delta r})}{\Delta r}\;. \tag{12}$$

We repeat this process for *ms_count* samples. For each sample we iterate through scattering count k until the next scattering event is more than *view_length* away from the central source(s) (twice half side length of the bounding box). In that case we consider the photon has escaped the medium.

[3] A comment on the *view_length* dependence here: in Eq. 10, the integral for the total scattering optical depth evaluates to infinity, and we estimate it by sampling one differential scattering optical depth, and multiplying a distance. We use $l = \mathit{view\_length}/2$ because this is the average randomly selected distance. Another choice could be to use $l = 0.661707\cdot\mathit{view\_length}$, which is the average distance between two randomly selected points within the bounding box. Any value that's proportional to *view_length* should be arguable, since the scattering path length in general agrees to the size of the object, which in turn agrees to an appropriately chosen *view_length* parameter. However, this approximation can create bias depending on where the photon is scattered. If the photon is scattered at the edge of the object and continues to travel out, the approximation drastically overestimates the path length. If the photon gets scattered inward the path length is underestimated. With large numbers of photons sampled the bias should be linear and dependent on l and how well *view_length* represents the scattering path lengths within the object.

3.4. *Peel-off*

At this stage, `siltsong` has produced an array representing a set of sources (central source(s), singly- or multiply-scattered light) at each position $\vec{p} = (px,\, py,\, d)$ within the medium. Sources located at greater depth d lie behind those in the foreground. We perform radiative transfer along the direction defined by each pixel $(px,\, py)$ in order to compute the specific intensity at depth $d = 0$ for each pixel. Starting from Eq. 2, we incorporate deposited scattered light sources when applicable:

$$dI_\nu(\vec{p}) = -I_\nu \kappa_\nu \, ds + S_\nu \kappa_{\nu,\,\text{non-scattering}} \, ds + \sum_{i=1}^{N_{ms}} \Delta I_{\nu,\,\text{dep},\,i} \, \delta^3(\vec{p} - \vec{p}_{\nu,\,\text{dep},\,i}) \ . \tag{13}$$

This expression is integrated in the simulation from the maximum depth to depth zero with $ds = view\_length/(depth \cdot depth\_substeps)$. The resulting $I_\nu(px,\, py)$ at $d = 0$ represents the specific intensity observed in the direction of pixel $(px,\, py)$ from a distant telescope. This final image can then be converted to units of photon flux or magnitudes, and/or convolved with a point spread function (PSF) as needed.