1 Second Order Approximation

The following approximation is used when calculating integrals (secord order Taylor expansion):

$$\begin{aligned} Psf(x,y) &\equiv \exp\left\{-\frac{1}{2}\left[S(x^2+y^2)+D(x^2-y^2)+2Kxy\right]\right\} \approx \\ &\approx k_{20}x^2+k_{11}xy+k_{02}y^2+k_{10}x+k_{01}y+k_{00} \qquad (1) \end{aligned}$$

Where for (x', y') being some point inside the region we will integrate over:

$$\begin{aligned} k_{20} &\equiv \frac{f_0}{2}\left\{\left[(S+D)x'+Ky'\right]^2-[S+D]\right\} && (2)\\ k_{11} &\equiv f_0\left\{\left[(S+D)x'+Ky'\right]\left[(S-D)y'+Kx'\right]-K\right\} && (3)\\ k_{02} &\equiv \frac{f_0}{2}\left\{\left[(S-D)y'+Kx'\right]^2-[S-D]\right\} && (4)\\ k_{10} &\equiv -f_0\left[(S+D)x'+Ky'\right] && (5)\\ k_{01} &\equiv -f_0\left[(S-D)y'+Kx'\right] && (6)\\ k_{00} &\equiv f_0 && (7)\\ f_0 &\equiv \exp\left\{-S(x'^2+y'^2)-D(x'^2-y'^2)-Kx'y'\right\} && (8) \end{aligned}$$

2 General Polynomial Expansion

A general expression can be derived for the polynomial coefficients in the Taylor expansion of the PSF function:

$$\begin{aligned} \frac{Psf(x+\delta x, y+\delta y)}{Psf(x,y)} &= \exp\Bigg\{-\frac{1}{2}\Big[(S+D)(2x\delta x+\delta x^2)+(S-D)(2y\delta y+\delta y^2)+ \\ &\qquad 2K(x\delta y+y\delta x+\delta x\delta y)\Big]\Bigg\} && (9)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\Big[(S+D)(2x\delta x+\delta x^2)+ \\ &\qquad +(S-D)(2y\delta y+\delta y^2)+ \\ &\qquad +2K(x\delta y+y\delta x+\delta x\delta y)\Big]^n && (10)\\ &= \sum_{i,j,k,l,m=0}^{\infty}\frac{(-1)^n}{2^n i!j!k!l!m!}C_{20}^i C_{11}^j C_{02}^k C_{10}^l C_{01}^m \delta x^{2i+j+l}\delta y^{2k+j+m} && (11) \end{aligned}$$

Where: $n=i+j+k+l+m$ and:

$$\begin{aligned} C_{20} &\equiv S+D && (12)\\ C_{11} &\equiv 2K && (13) \end{aligned}$$

$$\begin{aligned} C_{02} &\equiv S - D && (14)\\ C_{10} &\equiv 2\left[(S+D)x + Ky\right] && (15)\\ C_{01} &\equiv 2\left[(S-D)y + Kx\right] && (16) \end{aligned}$$

3 Imposing Limits on the Error in PSF Integrals

3.1 Constraining the Fractional Error in the Expansion

Let the difference between the approximation and the exact expression be denoted by Δ_2. Since a second order approximation is used, all terms satisfying $2(i+j+k)+l+m =< 2$ from Equation **??** are not present in Δ_2, while everything else remains. The following splitting is useful:

$$\begin{aligned} \frac{\Delta_2}{Psf} = & \sum_{2(j+k)+l+m>2}^{\infty} \frac{(-1)^n}{2^n j!k!l!m!} C_{11}^j C_{02}^k C_{10}^l C_{01}^m \delta x^{j+l} \delta y^{2k+j+m} - \\ & - \frac{C_{20}\delta x^2}{2} \sum_{2(j+k)+l+m>0}^{\infty} \frac{(-1)^n}{2^n j!k!l!m!} C_{11}^j C_{02}^k C_{10}^l C_{01}^m \delta x^{j+l} \delta y^{2k+j+m} \Big) + \\ & + \frac{\Delta_{i\geq 2}}{Psf} && (17) \\ = & \sum_{2k+l+m>2}^{\infty} \frac{(-1)^n}{2^n k!l!m!} C_{02}^k C_{10}^l C_{01}^m \delta x^l \delta y^{2k+m} - \\ & - \frac{C_{11}\delta x \delta y}{2} \sum_{2k+l+m>0}^{\infty} \frac{(-1)^n}{2^n k!l!m!} C_{02}^k C_{10}^l C_{01}^m \delta x^l \delta y^{2k+m} + \\ & + \frac{\Delta_{j\geq 2}}{Psf} - \\ & - \frac{C_{20}\delta x^2}{2} \sum_{2k+l+m>0}^{\infty} \frac{(-1)^n}{2^n k!l!m!} C_{02}^k C_{10}^l C_{01}^m \delta x^l \delta y^{2k+m} \Big) + \\ & - \frac{C_{20}\delta x^2}{2} \frac{\Delta_{j\geq 1}}{Psf} + \frac{\Delta_{i\geq 2}}{Psf} && (18) \\ = & \sum_{l+m>2}^{\infty} \frac{(-1)^n}{2^n l!m!} C_{10}^l C_{01}^m \delta x^l \delta y^m + \\ & - \frac{C_{02}\delta y^2}{2} \sum_{l+m>0}^{\infty} \frac{(-1)^n}{2^n l!m!} C_{10}^l C_{01}^m \delta x^l \delta y^m + \frac{\Delta_{k\geq 2}}{Psf} - \\ & - \frac{C_{11}\delta x\delta y + C_{20}\delta x^2}{2} \sum_{l+m>0}^{\infty} \frac{(-1)^n}{2^n l!m!} C_{10}^l C_{01}^m \delta x^l \delta y^m - \\ & - \frac{C_{11}\delta x\delta y + C_{20}\delta x^2}{2} \frac{\Delta_{k\geq 1}}{Psf} + \frac{\Delta_{j\geq 2}}{Psf} - \frac{C_{20}\delta x^2}{2} \frac{\Delta_{j\geq 1}}{Psf} + \frac{\Delta_{i\geq 2}}{Psf} && (19) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\Delta_{m\geq 3}}{Psf} - \frac{C_{10}\delta x}{2}\frac{\Delta_{m\geq 2}}{Psf} + \frac{C_{10}^2\delta x^2}{8}\frac{\Delta_{m\geq 1}}{Psf} + \frac{\Delta_{l\geq 3}}{Psf} \\
&\quad - \frac{C_{02}\delta y^2}{2}\frac{\Delta_{m\geq 1}}{Psf} - \frac{C_{02}\delta y^2}{2}\frac{\Delta_{l\geq 1}}{Psf} + \frac{\Delta_{k\geq 2}}{Psf} \\
&\quad - \frac{C_{11}\delta x\delta y + C_{20}\delta x^2}{2}\frac{\Delta_{m\geq 1}}{Psf} - \frac{C_{11}\delta x\delta y + C_{20}\delta x^2}{2}\frac{\Delta_{l\geq 1}}{Psf} - \\
&\quad - \frac{C_{11}\delta x\delta y + C_{20}\delta x^2}{2}\frac{\Delta_{k\geq 1}}{Psf} + \frac{\Delta_{j\geq 2}}{Psf} - \frac{C_{20}\delta x^2}{2}\frac{\Delta_{j\geq 1}}{Psf} + \frac{\Delta_{i\geq 2}}{Psf} \qquad (20) \\
&= \frac{\Delta_{m\geq 3}}{Psf} - \frac{C_{10}\delta x}{2}\frac{\Delta_{m\geq 2}}{Psf} + \\
&\quad + \left(\frac{C_{10}^2\delta x^2}{8} - \frac{C_{02}\delta y^2}{2} - \frac{C_{11}\delta x\delta y + C_{20}\delta x^2}{2}\right)\frac{\Delta_{m\geq 1}}{Psf} + \\
&\quad - \left(\frac{C_{02}\delta y^2 + C_{11}\delta x\delta y + C_{20}\delta x^2}{2}\right)\frac{\Delta_{l\geq 1}}{Psf} + \frac{\Delta_{l\geq 3}}{Psf} - \\
&\quad + \frac{\Delta_{k\geq 2}}{Psf} - \frac{C_{11}\delta x\delta y + C_{20}\delta x^2}{2}\frac{\Delta_{k\geq 1}}{Psf} + \frac{\Delta_{j\geq 2}}{Psf} - \frac{C_{20}\delta x^2}{2}\frac{\Delta_{j\geq 1}}{Psf} + \\
&\quad + \frac{\Delta_{i\geq 2}}{Psf} \qquad (21)
\end{aligned}
$$

Where:

$$
\begin{aligned}
\frac{\Delta_{i\geq 2}}{Psf} &\equiv \sum_{i=2}^{\infty}\sum_{j,k,l,m=0}^{\infty}\frac{(-1)^n}{2^n i!j!k!l!m!}C_{20}^i C_{11}^j C_{02}^k C_{10}^l C_{01}^m \delta x^{2i+j+l}\delta y^{2k+j+m} \qquad (22) \\
\frac{\Delta_{j\geq \mu}}{Psf} &\equiv \sum_{j=\mu}^{\infty}\sum_{k,l,m=0}^{\infty}\frac{(-1)^n}{2^n j!k!l!m!}C_{11}^j C_{02}^k C_{10}^l C_{01}^m \delta x^{j+l}\delta y^{2k+j+m} \qquad (23) \\
\frac{\Delta_{k\geq \mu}}{Psf} &\equiv \sum_{k=\mu}^{\infty}\sum_{l,m=0}^{\infty}\frac{(-1)^n}{2^n k!l!m!}C_{02}^k C_{10}^l C_{01}^m \delta x^{l}\delta y^{2k+m} \qquad (24) \\
\frac{\Delta_{l\geq \mu}}{Psf} &\equiv \sum_{l=\mu}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n}{2^n l!m!}C_{10}^l C_{01}^m \delta x^{l}\delta y^{m} \qquad (25) \\
\frac{\Delta_{m\geq \mu}}{Psf} &\equiv \sum_{m=\mu}^{\infty}\frac{(-1)^m}{2^m m!}C_{01}^m \delta y^{m} \qquad (26) \\
& \qquad (27)
\end{aligned}
$$

Considering each term separately:

$$
\begin{aligned}
\left|\frac{\Delta_{i\geq 2}}{Psf}\right| &= \left|\frac{C_{20}^2\delta x^4}{4}\sum_{i,j,k,l,m=0}^{\infty}\frac{(-1)^n}{2^n (i+2)!j!k!l!m!}C_{20}^i C_{11}^j C_{02}^k C_{10}^l C_{01}^m \delta x^{2i+j+l}\delta y^{2k+j+m}\right| \\
&= \frac{C_{20}^2\delta x^4}{4}\exp\left\{-\frac{1}{2}\left[C_{11}\delta x\delta y + C_{02}\delta y^2 + C_{10}\delta x + C_{01}\delta y\right]\right\}\left|\sum_{i=0}^{\infty}\frac{(-1)^i}{2^i (i+2)!}C_{20}^2\delta x^{2i}\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \quad \frac{C_{20}^2\delta x^4}{4}\exp\left\{-\frac{1}{2}\left[C_{11}\delta x\delta y + C_{02}\delta y^2 + C_{10}\delta x + C_{01}\delta y\right]\right\}\sum_{i=0}^{\infty}\frac{C_{20}^2\delta x^{2i}}{2^i(i+2)!} \\
&< \quad \left|\frac{C_{20}^2\delta x^4}{8}\right|\exp\left\{\frac{1}{2}\left[C_{02}\delta x^2 - C_{11}\delta x\delta y - C_{02}\delta y^2 - C_{10}\delta x - C_{01}\delta y\right]\right\} \qquad (28)
\end{aligned}
$$

Where δx and δy can be positive or negative. Similarly for the other terms, getting:

$$
\begin{aligned}
\frac{\Delta_{i\ge\mu}}{Psf} &< \left|\frac{C_{20}^{\mu}\delta x^{2\mu}}{2^{\mu}\mu!}\right|\zeta_{20}\zeta_{10} && (29)\\
\frac{\Delta_{j\ge\mu}}{Psf} &< \left|\frac{C_{11}^{\mu}\delta x^{\mu}\delta y^{\mu}}{2^{\mu}\mu!}\right|\zeta_{11}\zeta_{10} && (30)\\
\frac{\Delta_{k\ge\mu}}{Psf} &< \left|\frac{C_{02}^{\mu}\delta y^{2\mu}}{2^{\mu}\mu!}\right|\zeta_{02}\zeta_{10} && (31)\\
\frac{\Delta_{l\ge\mu}}{Psf} &< \left|\frac{C_{10}^{\mu}\delta x^{\mu}}{2^{\mu}\mu!}\right|\zeta_{10} && (32)\\
\frac{\Delta_{m\ge\mu}}{Psf} &< \left|\frac{C_{01}^{\mu}\delta y^{\mu}}{2^{\mu}\mu!}\right|\zeta_{01} && (33)
\end{aligned}
$$

Where:

$$
\begin{aligned}
\zeta_{01} &\equiv \exp\left\{\frac{|C_{01}\delta y|}{2}\right\} && (34)\\
\zeta_{10} &\equiv \exp\left\{\frac{|C_{10}\delta x|}{2}\right\}\zeta_{01} && (35)\\
\zeta_{02} &\equiv \exp\left\{\frac{|C_{02}\delta y^2|}{2}\right\} && (36)\\
\zeta_{11} &\equiv \exp\left\{\frac{|C_{11}\delta x\delta y|}{2}\right\}\frac{1}{\zeta_{02}} && (37)\\
\zeta_{20} &\equiv \exp\left\{\frac{|C_{20}\delta x^2|}{2}\right\}\zeta_{11} && (38)
\end{aligned}
$$

So finally we have:

$$
\begin{aligned}
\frac{|\Delta_2|}{Psf} \quad < \quad & \left\{\frac{\left|C_{01}^3\delta y^3\right|}{48} + \frac{\left|C_{10}C_{01}^2\delta x\delta y^2\right|}{16} + \right.\\
& \left. + \left|\frac{C_{10}^2\delta x^2}{4} - C_{02}\delta y^2 - \left(C_{11}\delta x\delta y + C_{20}\delta x^2\right)\right|\left|\frac{C_{01}\delta y}{4}\right|\right\}\zeta_{01} + \\
& \left\{\frac{\left|C_{02}\delta y^2 + C_{11}\delta x\delta y + C_{20}\delta x^2\right|\left|C_{10}\delta x\right|}{4} + \frac{\left|C_{10}^3\delta x^3\right|}{48}\right\}\zeta_{10} + \\
& + \left\{\frac{\left|C_{02}\delta y^2\right|\left|C_{11}\delta x\delta y + C_{20}\delta x^2\right|}{4} + \frac{C_{02}^2\delta y^4}{8}\right\}\zeta_{02}\zeta_{10} +
\end{aligned}
$$

$$+\left\{\frac{\left|C_{11}^2\delta x^2\delta y^2\right|}{8}+\frac{\left|C_{20}C_{11}\delta x^3\delta y\right|}{4}\right\}\zeta_{11}\zeta_{10}+\frac{\left|C_{20}^2\delta x^4\right|}{8}\zeta_{20}\zeta_{10} \quad (39)$$

3.2 Subdividing Pixels in Order to Achieve a Desired Precision in the Integral

In order to figure out how finely to subdivide a pixel in order to achive a prescribed precision, we will assume that the same number of subdividions (n) will be performed in the x as in the y direction.

If the actual integral of the PSF over a pixel is denoted by $I \equiv \int PSFdA$, we wish to split a pixel into enough parts that the overall approximation of the integral Q be within the less restrictive of some maximal fractional error ϵ_f and some maximal absolute error ϵ_a of I.

Since:

$$Error(\int PSFdA) = \int \Delta dA = \int \frac{\Delta}{PSF} PSFdA \leq \max\left|\frac{\Delta}{PSF}\right| I \quad (40)$$

Imposing a fractional error limit just means $\gamma < \epsilon_f$. Imposing an absolute error means that the sum of all absolute errors of each subdivision must be less than ϵ_a. One simple and not terribly bad way of achieving this is simply to require that the absolute error in each subdivision be no larger than ϵ_a/n^2.

Letting $\gamma \equiv \max\left|\frac{\Delta}{PSF}\right|$, we know

$$(1-\gamma)I < Q < (1+\gamma)I \quad \Longrightarrow \quad \frac{Q}{1+\gamma} < I < \frac{Q}{1-\gamma}$$

So requiring that the absolute error in a particular subdivision be no larger than ϵ_a/n^2 translates to:

$$\frac{\gamma}{1-\gamma} < \frac{\epsilon_a}{n^2 Q} \Longrightarrow \gamma < \frac{\frac{\epsilon_a}{n^2 Q}}{1+\frac{\epsilon_a}{n^2 Q}}$$

Where Q is the approximated integral in a subdivision.

From Section **??**, an upper limit as a function of the number of subdivisions can be written as:

$$\gamma < \frac{\alpha_1^{1/n}\alpha_2^{1/n^2}}{n^4} + \frac{\beta^{1/n}}{n^3} \quad (41)$$

With:

$$\begin{aligned}\beta \quad \equiv \quad & \left\{\frac{\left|C_{01}^3\delta y^3\right|}{48}+\frac{\left|C_{10}C_{01}^2\delta x\delta y^2\right|}{16}+\right.\\ & \left.+\left|\frac{C_{10}^2\delta x^2}{4}-C_{02}\delta y^2-\left(C_{11}\delta x\delta y+C_{20}\delta x^2\right)\right|\left|\frac{C_{01}\delta y}{4}\right|\right\}\zeta_{01}+\end{aligned}$$

$$\left\{\frac{\left|C_{02}\delta y^2+C_{11}\delta x\delta y+C_{20}\delta x^2\right|\left|C_{10}\delta x\right|}{4}+\frac{\left|C_{10}^3\delta x^3\right|}{48}\right\}\zeta_{10} \quad (42)$$

$$\alpha_1 \equiv \left\{\frac{\left|C_{02}\delta y^2\right|\left|C_{11}\delta x\delta y+C_{20}\delta x^2\right|}{4}+\frac{C_{02}^2\delta y^4}{8}+\frac{\left|C_{11}^2\delta x^2\delta y^2\right|}{8}+\right.$$
$$\left.\frac{\left|C_{20}C_{11}\delta x^3\delta y\right|}{4}+\frac{\left|C_{20}^2\delta x^4\right|}{8}\right\}\zeta_{10} \quad (43)$$

$$\alpha_2 \equiv \left\{\frac{\left|C_{02}\delta y^2\right|\left|C_{11}\delta x\delta y+C_{20}\delta x^2\right|}{4}+\frac{C_{02}^2\delta y^4}{8}\right\}\zeta_{02}+$$
$$+\left\{\frac{\left|C_{11}^2\delta x^2\delta y^2\right|}{8}+\frac{\left|C_{20}C_{11}\delta x^3\delta y\right|}{4}\right\}\zeta_{11}+\frac{\left|C_{20}^2\delta x^4\right|}{8}\zeta_{20} \quad (44)$$

Since subdividing a pixel into too many pieces at once will tend to severely overestimate the error, it is better to limit the number of subdivisions to some small number (n_{max}) and if more than that are required to consider each of them separately as a pixel and estimate how much furth to subdivide.

This leads to a simple scheme of starting with $n=1$ and incrementing n by one until Eq. **??** produces an upper limit less than $\max\left[\epsilon_f, \left(\frac{\epsilon_a}{n^2Q}\right)/\left(1+\frac{\epsilon_a}{n^2Q}\right)\right]$, or n_{max} is reached. If the first condition is met, then the integral is directly calculated on each subdivision, if the first condition is still not satisfied by $n=n_{max}$ the pixel is subdividid into $n_{max}\times n_{max}$ pieces and each piece is treated like a pixel, leading to further subdivisions.

3.3 Increasing the Expansion Order in order to Achieve a Desired Precision in the Integral

From Equation **??** we have:

$$\frac{Psf(x+\delta x,y+\delta y)}{Psf(x,y)}=\exp\left(-\frac{C_{20}\delta x^2}{2}\right)\exp\left(-\frac{C_{11}\delta x\delta y}{2}\right)\exp\left(-\frac{C_{02}\delta y^2}{2}\right)\exp\left(-\frac{C_{10}\delta x}{2}\right)\exp\left(-\frac{C_{01}\delta y}{2}\right) \quad (45)$$

If each of the terms above is split into some finite order polynomial approximation (S) and all remaining terms (Δ) we have:

$$\frac{Psf(x+\delta x,y+\delta y)}{Psf(x,y)}=(S_{20}+\Delta_{20})(S_{11}+\Delta_{11})(S_{02}+\Delta_{02})(S_{10}+\Delta_{10})(S_{01}+\Delta_{01}) \quad (46)$$

Where:

$$S_{20}\equiv\sum_{i=0}^{I}\frac{(-1)^iC_{20}^i\delta x^{2i}}{2^ii!} \quad , \quad \Delta_{20}\equiv\sum_{i=I+1}^{\infty}\frac{(-1)^iC_{20}^i\delta x^{2i}}{2^ii!} \quad (47)$$

$$S_{11}\equiv\sum_{j=0}^{J}\frac{(-1)^jC_{11}^j\delta x^j\delta y^j}{2^jj!} \quad , \quad \Delta_{11}\equiv\sum_{j=J+1}^{\infty}\frac{(-1)^jC_{11}^j\delta x^j\delta y^j}{2^jj!} \quad (48)$$

$$S_{02} \equiv \sum_{k=0}^{K} \frac{(-1)^k C_{02}^k \delta y^{2k}}{2^k k!} \quad , \quad \Delta_{02} \equiv \sum_{k=K+1}^{\infty} \frac{(-1)^k C_{02}^k \delta y^{2k}}{2^k k!} \tag{49}$$

$$S_{10} \equiv \sum_{l=0}^{L} \frac{(-1)^l C_{10}^l \delta x^l}{2^l l!} \quad , \quad \Delta_{10} \equiv \sum_{l=L+1}^{\infty} \frac{(-1)^l C_{10}^l \delta x^l}{2^l l!} \tag{50}$$

$$S_{01} \equiv \sum_{m=0}^{M} \frac{(-1)^m C_{01}^m \delta y^m}{2^m m!} \quad , \quad \Delta_{01} \equiv \sum_{m=M+1}^{\infty} \frac{(-1)^m C_{01}^m \delta y^m}{2^m m!} \tag{51}$$

$$\tag{52}$$

Since all the quantities approximated by the various S expansions are positive, with sufficiently high order approximation, all S expansions will also be positive. From this it follows that that the error in the PSF approximation (Δ_{IJKLM}) satisfies:

$$\begin{aligned} \frac{\Delta_{IJKLM}}{PSF} \leq\; & (S_{20} + |\Delta_{20}|)(S_{11} + |\Delta_{11}|)(S_{02} + |\Delta_{02}|)(S_{10} + |\Delta_{10}|)(S_{01} + |\Delta_{01}|) - \\ & - S_{20} S_{11} S_{02} S_{10} S_{01} \end{aligned} \tag{53}$$

So in order to derive an upper limit to the error in the PSF approximation we need only derive upper limits to each Δ quantity.

$$\begin{aligned} |\Delta_{20}| &= \left| \sum_{i=I+1}^{\infty} \frac{(-1)^i C_{20}^i \delta x^{2i}}{2^i i!} \right| \\ &= \frac{C_{20}^{I+1} \delta x^{2I+2}}{2^{I+1}} \left| \sum_{i=0}^{\infty} \frac{(-1)^i C_{20}^i \delta x^{2i}}{2^i (i+I+1)!} \right| \end{aligned} \tag{54}$$

We will now require that the expansion be of high enough order to satisfy: $2I + 2 > C_{20} \delta x^2$. Under this condition, the terms of the sum are monotonically decreasing. This means that if we substitute $(i + I + 1)!$ with $(I + 1)!i!$, the value of the sum will increase since even (positive) terms will be increased more than the subsequent odd (negative) terms. So we end up with:

$$\begin{aligned} |\Delta_{20}| &< \frac{C_{20}^{I+1} \delta x^{2I+2}}{2^{I+1} (I+1)!} \left| \sum_{i=0}^{\infty} \frac{(-1)^i C_{20}^i \delta x^{2i}}{2^i i!} \right| \\ &< \frac{C_{20}^{I+1} \delta x^{2I+2}}{2^{I+1} (I+1)!} \exp\left(-\frac{C_{20} \delta x^2}{2} \right) \end{aligned} \tag{55}$$

Similarly it follows:

$$|\Delta_{11}| < \left| \frac{C_{11} \delta x \delta y}{2} \right|^{J+1} \frac{1}{(J+1)!} \exp \left| \frac{C_{11} \delta x \delta y}{2} \right| \tag{56}$$

$$|\Delta_{02}| < \frac{C_{02}^{K+1} \delta y^{2K+2}}{2^{K+1} (K+1)!} \exp\left(-\frac{C_{02} \delta y^2}{2} \right) \tag{57}$$

$$|\Delta_{10}| \quad < \quad \left|\frac{C_{10}\delta x}{2}\right|^{L+1} \frac{1}{(L+1)!} \exp\left|\frac{C_{10}\delta x}{2}\right| \tag{58}$$

$$|\Delta_{01}| \quad < \quad \left|\frac{C_{01}\delta y}{2}\right|^{M+1} \frac{1}{(M+1)!} \exp\left|\frac{C_{01}\delta y}{2}\right| \tag{59}$$

The quantities Δ_{20} and Δ_{02} due to the required minimum expansion order are increasing functions of δx and δy respectively. The rest clearly are also. So a strict upper limit for the error in the integral can be found by using the largest by absolute value δx and δy in the Δ quantities and integrating the remaining S quantities in the expression for the error in the PSF estimation.

The expansion above is not a fixed order polynomial. Rather it independently controls the order of each term in the expansion, which might be somewhat inefficient, but otherwise the Δ and S quantities couple and strict limits to the integral are hard to derive in general.

4 The Following Most Probably Contains Many Errors

By similar logic to Section the error in a polynomial expansion of order up to N of the PSF satisfies:

$$\begin{aligned}
\frac{\Delta_N}{PSF} \quad = \quad & \frac{\Delta_{i>\frac{N}{2}}}{PSF} + \\
& + \sum_{i=0}^{\frac{N}{2}} \frac{(-1)^i C_{20}^i \delta x^{2i}}{2^i i!} \frac{\Delta_{j>\frac{N}{2}-i}}{PSF} + \\
& + \sum_{i=0}^{\frac{N}{2}} \frac{(-1)^i C_{20}^i \delta x^{2i}}{2^i i!} \sum_{j=0}^{\frac{N}{2}-i} \frac{(-1)^j C_{11}^j \delta x^j \delta y^j}{2^j j!} \frac{\Delta_{k>\frac{N}{2}-i-j}}{PSF} + \\
& + \sum_{i=0}^{\frac{N}{2}} \frac{(-1)^i C_{20}^i \delta x^{2i}}{2^i i!} \sum_{j=0}^{\frac{N}{2}-i} \frac{(-1)^j C_{11}^j \delta x^j \delta y^j}{2^j j!} \sum_{k=0}^{\frac{N}{2}-i-j} \frac{(-1)^k C_{02}^k \delta y^{2k}}{2^k k!} \frac{\Delta_{l>N-2i-2j-2k}}{PSF} + \\
& + \sum_{i=0}^{\frac{N}{2}} \frac{(-1)^i C_{20}^i \delta x^{2i}}{2^i i!} \sum_{j=0}^{\frac{N}{2}-i} \frac{(-1)^j C_{11}^j \delta x^j \delta y^j}{2^j j!} \sum_{k=0}^{\frac{N}{2}-i-j} \frac{(-1)^k C_{02}^k \delta y^{2k}}{2^k k!} \times \\
& \qquad \times \sum_{l=0}^{N-2(i+j+k)} \frac{(-1)^l C_{10}^l \delta x^l}{2^l l!} \frac{\Delta_{m>N-2(i+j+k)-l}}{PSF}
\end{aligned} \tag{60}$$

Actually a more stringent limit can be derived by writing directly the expansion of the integral over the range $x-\delta x < x < x+\delta x$, $y-\delta y < y < y+\delta y$:

$$I = 4f_0\delta x\delta y \sum_{\substack{j+l\,:\,\text{even}\\ j+m\,:\,\text{even}}} \frac{(-1)^n C_20^i C_{11}^j C_{02}^k C_{10}^l C_{01}^m}{2^n i!j!k!l!m!} \frac{\delta x^{2i+j+l}\delta y^{2k+j+m}}{(2i+j+l+1)(2k+j+m+1)} \tag{61}$$

If we estimate the sum by only including $i <= I$, $j <= J$, $k <= K$, $l <= L$ and $m <= M$, the following quantities can be used to calculate a strict upper limit on the error made in the estimation:

$$S_p \equiv \sum_{i=0}^{p_0 I}\sum_{j=0}^{p_1 J}\sum_{k=0}^{p_2 K}\sum_{l=0}^{p_3 L}\sum_{m=0}^{p_4 M} \frac{(-1)^n C_20^i C_{11}^j C_{02}^k C_{10}^l C_{01}^m}{2^n i!j!k!l!m!}\delta x^{2i+j+l}\delta y^{2k+j+m} R_1 R_2 \tag{62}$$

$$\Delta_0 \equiv \frac{C_{20}^{I+1}\delta x^{2I+2}}{2^{I+1}(2I+3)(I+1)!}\exp\left\{\frac{C_{20}\delta x^2}{2}\right\} \tag{63}$$

$$\Delta_1 \equiv \frac{\left|C_{11}^{J+1}\right|\delta x^{J+1}\delta y^{J+1}}{2^{J+1}(J+2)(J+2)!}\exp\left\{\frac{|C_{11}|\,\delta x\delta y}{2}\right\} \tag{64}$$

$$\Delta_2 \equiv \frac{C_{02}^{K+1}\delta y^{2K+2}}{2^{K+1}(2K+3)(K+1)!}\exp\left\{\frac{C_{02}\delta y^2}{2}\right\} \tag{65}$$

$$\Delta_3 \equiv \frac{\left|C_{10}^{L+1}\delta x^{L+1}\right|}{2^{L+1}(L+2)!}\exp\left\{\frac{C_{10}\delta x}{2}\right\} \tag{66}$$

$$\Delta_4 \equiv \frac{\left|C_{01}^{M+1}\delta y^{M+1}\right|}{2^{M+1}(M+2)!}\exp\left\{\frac{C_{01}\delta y}{2}\right\} \tag{67}$$

with:

$$R_1 \equiv \begin{cases} \frac{1-(j+l)\%2}{(2i+j+l+1)} & \text{if} \quad p_0=p_1=p_3=1 \\ 1 & \text{otherwise} \end{cases} \tag{68}$$

$$R_2 \equiv \begin{cases} \frac{1-(j+m)\%2}{(2k+j+m+1)} & \text{if} \quad p_1=p_2=p_4=1 \\ 1 & \text{otherwise} \end{cases} \tag{69}$$

Above p is a vector of 5 values each of which can be either 0 or 1.

A strict upper limit to the error made by estimating the value of the integral by only including the specified terms is given by:

$$\Delta I < 4f_0\delta x\delta y \sum_{p\neq\overrightarrow{1}} S_p \prod_{n=0}^{4}(\Delta_n)^{1-p_n} \tag{70}$$

And the above sum with $p=\overrightarrow{1}$ is the estimate for I.

If instead of imposing independent limits on each index we wish to impose a limit on the overall order (2N), the error in the integral satisfies:

$$\Delta I < 4f_0\delta x\delta y\Bigg(\quad \Delta_0(N+1)\Delta_1(0)\Delta_2(0)\Delta_3(0)\Delta_4(0) +$$

$$\sum_{i=0}^{N} \frac{(-1)^i C_{20}^i \delta x^{2i}}{2^i i!} \Delta_1(N-i+1)\Delta_2(0)\Delta_3(0)\Delta_4(0) +$$

$$\sum_{i=0}^{N} \sum_{j=0}^{N-i} \frac{(-1)^{i+j} C_{20}^i C_{11}^j \delta x^{2i+j} \delta y^j}{2^{i+j} i! j!} \Delta_2(N-i-j+1)\Delta_3(0)\Delta_4(0) +$$

$$\sum_{i=0}^{N} \sum_{j=0}^{N-i} \sum_{k=0}^{N-i-j} \frac{(-1)^{i+j+k} C_{20}^i C_{11}^j C_{20}^k \delta x^{2i+j} \delta y^{2k+j}}{2^{i+j+k} i! j! k!} \Delta_3(N-i-j-k+1)\Delta_4(0) +$$

$$\sum_{i=0}^{N} \sum_{j=0}^{N-i} \sum_{k=0}^{N-i-j} \sum_{l=0}^{N-i-j-k} \frac{(-1)^{i+k} C_{20}^i C_{11}^j C_{20}^k \delta x^{2i+j} \delta y^{2k+j}}{2^{i+j+k} i! j! k!} \Delta_3(N-i-j-k+1)\Delta_4(0) +$$

$$\Bigg) \tag{71}$$