

FINITE GROUPS WITH TWO CONJUGACY CLASSES OF p -ELEMENTS AND RELATED QUESTIONS FOR p -BLOCKS

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ABSTRACT. We prove that a finite group in which any two nontrivial p -elements are conjugate have Sylow p -subgroups which are either elementary abelian or extraspecial of order p^3 and exponent p .

1. INTRODUCTION

Let G be a finite group and let p be a prime. The main result of this note is to solve a group theoretical problem that naturally arises in block theory (and in the theory of fusion systems).

Theorem A. *Let p be a prime and G a finite group in which any two nontrivial p -elements are conjugate. Then one of the following holds:*

- (i) *The Sylow p -subgroups of G are elementary abelian;*
- (ii) *$p = 3$ and $\mathbf{O}^{p'}(G/\mathbf{O}_{p'}(G))$ is isomorphic to Ru , J_4 or ${}^2F_4(q)'$ with $q = 2^{6b\pm 1}$ for a nonnegative integer b ;*
- (iii) *$p = 5$ and $G/\mathbf{O}_{p'}(G)$ is isomorphic to Th .*

In cases (ii) and (iii), the Sylow p -subgroups of G are extra-special of order p^3 and exponent p .

Theorem A can be reformulated in terms of Brauer blocks. If B is a p -block of a finite group G , recall that a B -subsection is a pair (u, b_u) consisting of a p -element $u \in G$ and a p -block b_u of the centralizer $\mathbf{C}_G(u)$ such that the induced block $(b_u)^G = B$. The *trivial* B -subsection is $(1, B)$. It is a natural problem to study blocks in which all the non-trivial subsections are G -conjugate, and the question is whether or not in this case the defect groups of B should be elementary abelian or extra-special of order p^3 and exponent p . Theorem A provides an answer to this for the principal block of G . (In Section 3 of this paper, we shall comment on the corresponding question for fusion systems.)

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Theorem A is also related to a recent theorem due to the second and fourth authors of this paper, which states that for $p \neq 3, 5$, a finite group G has abelian Sylow p -subgroups if and only if the conjugacy class size of every p -element of G is not divisible by p , cf. [23]. This group theoretical result also admits a block theoretical reformulation which we find of interest. We discuss this and some other related problems in Section 3 of this paper. Finally, in Section 4, we study blocks with a small number of irreducible characters, a topic that naturally connects with the previous parts of the paper.

2. PROOF OF THEOREM A

We begin by proving a slightly more general result than Theorem A.

Theorem 2.1. *Let p be a prime and G a finite group in which any two nontrivial cyclic p -subgroups are conjugate. Then one of the following holds:*

- (i) *The Sylow p -subgroups of G are elementary abelian;*
- (ii) *$p = 3$ and $\mathbf{O}_{p'}(G/\mathbf{O}_{p'}(G))$ is isomorphic to Ru , J_4 or ${}^2F_4(q)'$ with $q = 2^{6b \pm 1}$ for a nonnegative integer b ;*
- (iii) *$p = 5$ and $G/\mathbf{O}_{p'}(G)$ is isomorphic to Th .*

As might be expected, the proof of Theorem 2.1 uses the Classification of Finite Simple Groups.

Note that Theorem 2.1 is trivial in the case where p does not divide $|G|$. So in proving Theorem 2.1 we will assume that $|G|$ is divisible by p . Then the assumptions in Theorem 2.1 imply that the exponent of P , $\exp(P)$, is p if $P \in \text{Syl}_p(G)$. In particular, if $p = 2$, then certainly P is elementary abelian. Furthermore we have that $|g^G|$ has the same size for all $1 \neq g \in P$. By choosing $g \in \mathbf{Z}(P)$, we have that this common size is coprime to p . Hence, if $p > 5$ then Theorem 2.1 follows from the main result of [23]. So in what follows we may assume that $p = 3$ or 5 .

Next we prove Theorem 2.1 for finite simple groups.

Theorem 2.2. *Let S be a finite simple group and let $p \in \{3, 5\}$ be a prime divisor of $|S|$. Suppose that $|g^S|$ is the same (and hence coprime to p) for all the nontrivial p -elements $g \in S$, and moreover assume that $\exp(P) = p$ for $P \in \text{Syl}_p(S)$. Then one of the following holds:*

- (i) *The Sylow p -subgroups of S are elementary abelian.*
- (ii) *$P \cong p_+^{1+2}$ is extra-special of order p^3 . Furthermore, either $p = 3$ and S is isomorphic to Ru , J_4 or ${}^2F_4(q)'$ with $q = 2^{6b \pm 1}$ for a nonnegative integer b , or $p = 5$ and $S \cong Th$.*

Lemma 2.3. *Theorem 2.2 holds if S is an alternating or sporadic simple group.*

Proof. Repeat the proof of [23, Lemma 2.2]. □

Lemma 2.4. *Theorem 2.2 holds if S is a finite simple group of Lie type in characteristic p .*

Proof. It is convenient to view S as $[H, H]$, where $H = \mathcal{G}^F$ for a simple algebraic group \mathcal{G} of adjoint type and a Frobenius map $F : \mathcal{G} \rightarrow \mathcal{G}$; in particular, $\mathbf{Z}(\mathcal{G}) = 1$ and $|H : S|$ is coprime to p . Suppose first that p is a good prime for \mathcal{G} . Then the proof of [23, Lemma 2.3] shows that \mathcal{G} is of type A_1 , i.e. $S = PSL_2(q)$ and so its Sylow p -subgroups are abelian.

Assume now that p is a bad prime for \mathcal{G} ; in particular, \mathcal{G} is exceptional since $p > 2$. Suppose that \mathcal{G} is of type G_2 , and so $p = 3$. If $H = {}^2G_2(q)$, by [19, Table 22.2.7] we can find a p -element $u \in H$ such that $|\mathbf{C}_H(u)| = 2q^2$, whence $q || u^H$. Similarly, if $H = G_2(q)$, by [19, Table 22.2.6] we can find a p -element $u \in H$ such that $|\mathbf{C}_H(u)| = 2q^4$, whence $q^2 || u^H$. If $H = F_4(q)$ (and $p = 3$), by [19, Table 22.2.4] we can find a p -element $u \in H$ such that $|\mathbf{C}_H(u)| = 2q^6$, whence $q^6 || u^H$. If $H = E_6(q)$ or ${}^2E_6(q)$, by [19, Table 22.2.3] we can find a p -element $u \in H$ such that $|\mathbf{C}_H(u)| = q^8$, whence $q^{28} || u^H$. If $H = E_7(q)$, by [19, Table 22.2.2] we can find a p -element $u \in H$ such that $|\mathbf{C}_H(u)| = 2q^{21}$, whence $q^{42} || u^H$. Finally, if $H = E_8(q)$, by [19, Table 22.2.1] we can find a p -element $u \in H$ such that $|\mathbf{C}_H(u)| = 2q^{28}$, whence $q^{92} || u^H$. \square

In what follows, we use the notation SL^ϵ to denote SL when $\epsilon = +$ or $+1$, and SU when $\epsilon = -$ or -1 , and similarly for the Lie types A_n^ϵ . Furthermore, N_p denotes the p -part of the integer N . We use the notation for various finite classical groups as described in [16].

Lemma 2.5. *Theorem 2.2 holds if the simple group S is a finite classical group in characteristic $r \neq p$.*

Proof. (i) First we consider the case $S = PSL_n^\epsilon(q)$, where $q = r^f$ and $\epsilon = \pm 1$. Since $p \neq r$ divides $|S|$, there is a smallest positive integer $m \leq n$ such that $p|(q^m - \epsilon^m)$. View $S = L/\mathbf{Z}(L)$, with $L = SL_n^\epsilon(q)$. Now the proof of [23, Proposition 2.5] shows that $m = 1$ and $n = p = 3$. Suppose that $9|(q - \epsilon)$. Choosing $\theta \in \overline{\mathbf{F}}_q$ of order 9 and $g = \text{diag}(\theta, \theta^{-1}, 1) \in L$, we see that g has order 9 in S , contradicting the condition that $\exp(P) = p$. On the other hand, if $9 \nmid (q - \epsilon)$, then P is elementary abelian of order 9.

(ii) For all other classical groups, since $p > 2$ we may without any loss replace S by $G = Sp_{2n}(q)$, $GO_{2n+1}(q)$, $GO_{2n}^+(q)$, or $GO_{2n}^-(q)$, if S is of type C_n , B_n , D_n , or 2D_n , respectively. In all these cases, there is a smallest positive integer $m \leq n$ such that $p|(q^{2m} - 1)$, and $\alpha = \pm 1$ such that $p|(q^m - \alpha)$. Write $(q^m - \alpha)_p = p^c$. Since $p > 2$ we have that $(q^{2mp} - 1)_p = (q^{mp} - \alpha)_p = p^{c+1}$. If $n < mp$, then the proof of [23, Proposition 2.5] shows that the Sylow p -subgroups of S are abelian. So we may assume that $n \geq mp$.

Suppose first that $(q, m, p, \alpha) = (2, 1, 3, -)$. In this case $Sp_6(2)$ embeds in G and so P contains an element of order 9, a contradiction.

Hence we may assume that $(q, m, p, \alpha) \neq (2, 1, 3, -)$. Then the proof of [23, Proposition 2.5] shows that G contains an element of order $p^{c+1} > p$, unless $G = GO_{mp}^{-\alpha}(q)$, in which case the Sylow p -subgroups of S are abelian. \square

Lemma 2.6. *Theorem 2.2 holds if S is an exceptional finite simple group of Lie type in characteristic $r \neq p$.*

Proof. (i) The case $S = {}^2F_4(2)'$ can be checked directly using [4], so we will assume $S \not\cong {}^2F_4(2)'$. We view $S = G/\mathbf{Z}(G)$, where $G = \mathcal{G}^F$ for a simple algebraic group \mathcal{G} of simply connected type and a Frobenius map $F : \mathcal{G} \rightarrow \mathcal{G}$. If $G \in \{{}^2B_2(q), {}^2G_2(q)\}$ is a Suzuki or a Ree group, then the Sylow p -subgroups of G are all abelian as $p > 2$. The same is true when $G = S = {}^2F_4(q)$ with $q \geq 8$, unless $p = 3$. In the latter case, $p = 3|(q+1)$; moreover, according to [18, Table 5.1], G contains a subgroup $X \cong SU_3(q)$. If $9|(q+1)$, then, as we noted in the proof of Lemma 2.5, X contains an element of order 9, contrary to $\exp(P) = 3$. Assume that $9 \nmid (q+1)$, which means that $q = 2^{6b \pm 1}$ for some $b \geq 1$. Then $|S|_3 = 27$, and P can be embedded in ${}^2F_4(2)'$, whence $P \cong 3_+^{1+2}$.

(ii) In all the other cases, we can write $|G| = q^a \cdot \prod_i \Phi_{m_i}^{b_i}(q)$ for some power $q = r^f$ of r , $a, b_i > 0$, and $m_i > 0$ pairwise distinct, where $\Phi_m(t)$ is the m^{th} cyclotomic polynomial in t . According to [9, §4.10.2], if p divides exactly one $\Phi_{m_i}(q)$, then the Sylow p -subgroups of G are abelian (in fact they are homocyclic of rank b_i). We will now assume that p divides $\Phi_{m_i}(q)$ for more than one m_i .

Here we consider the case $G = S = G_2(q)$. The above condition on p implies that there is some $\epsilon = \pm 1$ such that $p = 3|(q - \epsilon)$. Then, by [18, Table 5.1], $S > X \cong SL_3^\epsilon(q)$. Now if $9|(q - \epsilon)$, then, as noted above, X and S contain an element of order 9. More generally, according to [20], S contains an element a of order 3, with centralizer of type $A_1(q) \cdot C_{q-\epsilon}$, whence $|\mathbf{C}_S(a)|_3 = (q - \epsilon)_3^2$ and $3||a^S|$.

Similarly, if $G = S = {}^3D_4(q)$, then again $p = 3|(q - \epsilon)$ for some $\epsilon = \pm 1$. Then, according to [20], S contains an element a of order 3, with centralizer of type $A_1(q^3) \cdot C_{q-\epsilon}$, whence $|\mathbf{C}_S(a)|_3 = 3(q - \epsilon)_3^2$ and $3||a^S|$.

Assume that $p = 3$ and S is one of the remaining exceptional groups. Then again $p|(q - \epsilon)$ for some $\epsilon = \pm 1$, and S contains a subquotient $X \cong \Omega_9(q)$. As mentioned in the proof of Lemma 2.5, then X contains an element of order 9, a contradiction.

(iii) It remains to consider the case $p = 5$ and $p|\Phi_{m_i}(q)$ for $i = 1, 2$ and $m_1 \neq m_2$. This happens precisely when

- (a) $5|(q - \epsilon)$ for some $\epsilon = \pm 1$, and $G \in \{E_6^\pm(q), E_7(q), E_8(q)\}$, or
- (b) $5|(q^2 + 1)$ and $G = E_8(q)$.

In the case of (a), note that G contains a subgroup $X \cong SL_6^\epsilon(q)$, according to [18, Table 5.1]. In turn, X contains $Y \cong GL_5^\epsilon(q) > C_{q^5 - \epsilon}$ and so it contains an element of order 25, a contradiction.

Suppose we are in the case of (b). According to [18, Table 5.1], $G = S$ contains a maximal subgroup $X \cong SU_5(q^2) \cdot C_4$. Now if $25|(q^2 + 1)$, then X and G contain an element of order 25, a contradiction. In general, we can find an element $z \in \mathbf{Z}(X')$ of order 5. We claim that $C := \mathbf{C}_G(z) \leq X$. (Indeed, $Z := \mathbf{Z}(X') = \langle z \rangle$ is normalized by X which is maximal in G , whence $\mathbf{N}_G(Z) = X$ or $Z \triangleleft G$. In the latter case, since G is perfect, we would have that $Z \leq \mathbf{Z}(G) = 1$, a contradiction. Hence $X = \mathbf{N}_G(Z) \geq \mathbf{C}_G(z)$.) On the other hand, also according to [18, Table 5.1], G contains a subgroup Y of type $C_d^2 \cdot (P\Omega_8^+(q))^2$, where $d = \gcd(2, q - 1)$. Choosing an element of order 5 lying in one factor $P\Omega_8^+(q)$, we see that G contains an element $y \in Y$ of order 5 such that $\mathbf{C}_G(y)$ contains a subquotient $\cong P\Omega_8^+(q)$. In particular, $|\mathbf{C}_G(y)|$ is divisible by $q^3 - 1$. But obviously $|X|$ is not divisible by $q^3 - 1$. It follows that $|y^G| \neq |z^G|$, a contradiction. \square

Proof of Theorem 2.1. As mentioned above, by the main result of [23], we may assume that $p \in \{3, 5\}$. Suppose that G satisfies the hypothesis of Theorem 2.1 and that $P \in \text{Syl}_p(G)$ is non-abelian.

Step 1. We have that $\mathbf{O}_p(G) = 1$ and may also assume that $\mathbf{O}_{p'}(G) = 1$.

Indeed, suppose $Q := \mathbf{O}_p(G) \neq 1$. Then $1 \neq \mathbf{Z}(Q) \leq P$. By the assumption, every nontrivial cyclic p -subgroup of P is conjugate to some subgroup of $\mathbf{Z}(Q) \triangleleft G$, whence $\mathbf{Z}(Q) = P$ and P is abelian. So we have that $\mathbf{O}_p(G) = 1$. Next, by Sylow's Theorem, two non-cyclic p -subgroups of G are conjugate in G if and only if their images are conjugate in $G/\mathbf{O}_{p'}(G)$. Furthermore, the Sylow p -subgroups of $G/\mathbf{O}_{p'}(G)$ are isomorphic to P , so they are not abelian. Replacing G by $G/\mathbf{O}_{p'}(G)$, we may assume that $\mathbf{O}_{p'}(G) = 1$.

Step 2. We have that G has a unique minimal normal subgroup K , which is non-abelian of order divisible by p .

Suppose that K_i are two distinct minimal normal subgroups of G , where $i = 1, 2$. Hence $1 = K_1 \cap K_2$. Now, by Step 1 we know that K_i contains nontrivial cyclic p -subgroups Q_i for $i = 1, 2$ as $\mathbf{O}_{p'}(G) = 1$, and Q_1 and Q_2 are not G -conjugate. Hence, G has a unique minimal normal subgroup K of order divisible by p . Now if K is abelian, then $\mathbf{O}_p(G) \neq 1$, contrary to Step 1.

Step 3. We have that K is simple non-abelian and $K = \mathbf{O}^{p'}(G)$. Furthermore, K satisfies the assumptions of Theorem 2.2.

Since K is minimal normal in G , we see that $K = S_1 \times \cdots \times S_n \cong S^n$, where the S_i 's are non-abelian simple groups of order divisible by p which are transitively permuted by G .

Suppose that $n > 1$. Then we can find a p -element $1 \neq x \in S$ and consider the p -elements

$$y = (x, 1, \dots, 1), \quad z = (x, x, 1, \dots, 1)$$

in K . Now the cyclic p -subgroups $\langle y \rangle$ and $\langle z \rangle$ are not G -conjugate, a contradiction.

Thus K is simple non-abelian. The uniqueness of K implies that $\mathbf{C}_G(K) = 1$ and so G embeds in $\text{Aut}(K)$. Since all cyclic p -subgroups of G are conjugate to a subgroup of K , we see that $K \geq \mathbf{O}^{p'}(G)$, whence $K = \mathbf{O}^{p'}(G)$ by the minimality of K .

It is obvious that $\exp(P) = p$. Moreover, for any two nontrivial p -elements in K , their centralizers in G are G -conjugate. Hence also their centralizers in K are G -conjugate; in particular, these K -centralizers have the same order.

Step 5. Now we apply Theorem 2.2 to $K = \mathbf{O}^{p'}(G)$. Note that in the case $p = 5$ and $K \cong Th$, $G = K$ since $G \hookrightarrow \text{Aut}(K) \cong K$. Also, the Sylow p -subgroups of K are isomorphic to P and so are non-abelian. Now we see that G satisfies the conclusions (ii) or (iii) of Theorem 2.1 and so we are done. \square

3. RELATED QUESTIONS ON BLOCKS AND FUSION SYSTEMS

In this section we ask some related questions and prove a few results. If B is a p -block of G , recall that a B -subsection (u, b_u) is called *major* if the defect groups of b_u are also defect groups of B . Inspired by the main result of [23], we propose the following.

Question 3.1. *Let B be a p -block of a finite group, and suppose that all B -subsections are major. What can be said about the structure of the defect groups of B ?*

The main result of [23] provides an answer to Question 3.1 whenever $3 \neq p \neq 5$ and B is the principal block: the defect groups of B in this case should be abelian. Another partial answer is given by the following result which we prove below.

Theorem 3.2. *Let B be a p -block of a finite p -solvable group G . Then all B -subsections are major if and only if the defect groups of B are abelian.*

Theorem 3.2 generalizes several results in the literature, such as Lemma 4.1 in [21] and parts of the main theorem of [28]. J. Olsson [25] already observed that Theorem 3.2 does not hold for arbitrary groups G .

It is well-known that the fusion system of a p -solvable group on its Sylow p -subgroup is p -solvable (cf. p. 2446 of [5]) and that saturated subsystems of p -solvable fusion systems are again p -solvable (cf. p. 2431 of [5]). Thus the fusion system of a p -block of a p -solvable group on its defect group is p -solvable and therefore constrained. (For background on fusion systems we refer to [1] and [6].) Thus Theorem 3.2 is a consequence of the following result on fusion systems. We recall that elements $x, y \in P$ are called \mathcal{F} -conjugate if $f(x) = y$ for a morphism $f : \langle x \rangle \rightarrow \langle y \rangle$ in \mathcal{F} .

Theorem 3.3. *Let \mathcal{F} be a constrained fusion system on a finite p -group P . Suppose that every element in P is \mathcal{F} -conjugate to an element in $\mathbf{Z}(P)$. Then P is abelian.*

Proof. By Theorem 2.5 in [5], there exists a finite group G with Sylow p -subgroup P such that $\mathbf{O}_{p'}(G) = 1$, $\mathbf{C}_G(\mathbf{O}_p(G)) \subseteq \mathbf{O}_p(G)$ and $\mathcal{F} = \mathcal{F}_P(G)$. Let $Q := \mathbf{O}_p(G)$ and $x \in P$. Then our hypothesis implies that there exists an element $g \in G$ such that

$$gxg^{-1} \in \mathbf{Z}(P) \leq \mathbf{C}_G(Q) \leq \mathbf{Z}(Q)$$

which is normal in G , and therefore $x \in \mathbf{Z}(Q)$. This shows that $P = \mathbf{Z}(Q)$; in particular, P is abelian. \square

Next we ask whether the following generalization of Theorem A holds:

Question 3.4. *Let B be a p -block of a finite group G , and suppose that all nontrivial B -subsections are conjugate in G . Are the defect groups of B necessarily either elementary abelian or extraspecial of order p^3 and exponent p ?*

Since again Question 3.4 is mainly concerned with the fusion system of B we may formulate an even more general question for fusion systems:

Question 3.5. *Let \mathcal{F} be a saturated fusion system on a finite p -group P , and suppose that any two nontrivial cyclic subgroups of P are \mathcal{F} -conjugate. Is P necessarily abelian or extraspecial of order p^3 and exponent p ?*

If P is extraspecial of order p^3 and exponent p then the saturated fusion systems on P are described in [27]. It follows from these results that $p \in \{3, 5, 7\}$ whenever any two nontrivial elements in P are \mathcal{F} -conjugate. Also, if B is a p -block of a finite group with extraspecial defect groups of order p^3 and exponent p such that any two nontrivial B -subsections are conjugate, then the results in [27] and [14] imply that $p \in \{3, 5\}$.

There are several ways to measure the complexity of a block, including to study the structure of a defect group D of B , or the quantities $k(B)$ or $l(B)$. (As usual, here $k(B) = |\text{Irr}(B)|$ and $l(B) = |\text{IBr}(B)|$ where $\text{Irr}(B)$ denotes the set of irreducible ordinary characters of G associated to B , and $\text{IBr}(B)$ denotes the set of irreducible Brauer characters of G associated to B .) Another interesting invariant of the block B is the difference $k(B) - l(B)$. It was already known by Richard Brauer that $k(B) - l(B) = 0$ if and only if $D = 1$ (if and only if $k(B) = l(B) = 1$). From this point of view, the next natural step seems to be to study blocks with

$$k(B) - l(B) = 1,$$

which is one of the problems that led us to the main result of this paper. An important formula by Brauer shows that every p -block B of a finite group G satisfies

$$k(B) = \sum_{(u, b_u)} l(b_u)$$

where (u, b_u) ranges over a transversal for the conjugacy classes of B -subsections.

In particular, Brauer's formula implies that, for a p -block B with $k(B) - l(B) = 1$, all nontrivial B -subsections are conjugate. In this case, a result by Fujii [7] shows that the Cartan matrix of B has determinant $|D|$. Moreover, the stable center $\bar{Z}(B) := Z(B)/Z^{\text{pr}}(B)$ is a symmetric algebra, by the results in [13]. In fact, the stable center of B has dimension 2. Theorem 1.1 in [13] implies that, if B is the principal block of G and $k(B) - l(B) = 1$ then the Sylow p -subgroups of G are elementary abelian. Next, we prove more generally the following result.

Theorem 3.6. *Let B be a p -block of a finite group G with $k(B) - l(B) = 1$. Suppose that the fusion system of B is nonexotic (for instance, if B is the principal block or if G is p -solvable). Then the defect groups of B are elementary abelian.*

Recall that a fusion system \mathcal{F} on a finite p -group P is called *nonexotic* if there exists a finite group X with Sylow p -subgroup P such that \mathcal{F} is the fusion system $\mathcal{F}_P(X)$ on P coming from X . (Otherwise the fusion system \mathcal{F} is called *exotic*.) It seems to be an open question whether all fusion systems coming from blocks are nonexotic. A negative answer to Question 3.4 would provide an example of an exotic block fusion system.

Proof of Theorem 3.6. Assume that B has a nonabelian defect group D . Since the fusion system \mathcal{F} of B on D is nonexotic, Theorem A implies that $p \in \{3, 5\}$ and that D is extraspecial of order p^3 and exponent p .

Suppose first that $p = 5$. Then \mathcal{F} is the fusion system of the sporadic simple Thompson group Th on its Sylow 5-subgroup. Now Proposition 6.1 in [11] shows that B is Morita equivalent to the principal 5-block B_0 of Th . In particular, we have $k(B_0) - l(B_0) = 1$. Let (u, b_u) be a nontrivial B_0 -subsection. Then b_u is the principal 5-block of $\mathbf{C}_{Th}(u)$, and $l(b_u) = 1$. Thus $\mathbf{C}_{Th}(u)$ is 5-nilpotent, a contradiction.

It remains to consider the case $p = 3$. Let (u, b_u) denote a nontrivial B -subsection, and denote by \bar{b}_u the unique 3-block of $\mathbf{C}_G(u)/\langle u \rangle$ dominated by b_u . Then $1 = l(b_u) = l(\bar{b}_u)$, and \bar{b}_u has an elementary abelian defect group of order 9. By Theorem A, we may assume that \mathcal{F} is the fusion system of ${}^2F_4(2)'$ or J_4 on its Sylow 3-subgroup. Thus the inertial quotient of B is isomorphic to D_8 or SD_{16} respectively, by the results in [27] (cf. [22]). It follows easily that the inertial quotient of \bar{b}_u is isomorphic to C_4 or Q_8 respectively. However, if the inertial quotient of \bar{b}_u is isomorphic to C_4 , then the results of [15] lead to the contradiction $l(\bar{b}_u) = 4$.

Thus we may assume that the inertial quotient of \bar{b}_u is isomorphic to Q_8 . Then the arguments in [15] show that there are only two \bar{b}_u -subsections, and we obtain $k(\bar{b}_u) = 2$. However, then the defect groups of \bar{b}_u have order 2, a contradiction. \square

Examples of p -blocks B with $k(B) - l(B) = 1$ are the p -blocks of multiplicity one introduced in [21] by G. Michler. We recall that the *multiplicity* $\mu(B)$ of a p -block B

is defined by

$$\mu(B) := \max\{c_{ii} - 1 : i = 1, \dots, l(B)\}$$

where $C = (c_{ij})$ denotes the Cartan matrix of B . Thus $\mu(B) = 1$ if and only if all diagonal entries in the Cartan matrix of B are equal to 2.

In [21], Michler showed that a block B of multiplicity one satisfies $k(B) - l(B) = 1$, and that all irreducible ordinary characters associated to B have height zero. Thus, Brauer's Height Zero Conjecture predicts that B should have abelian defect groups. In fact, in [21], it is shown that a p -block B of a finite group G with multiplicity one has abelian defect groups if G is p -solvable or if $p = 2$.

The following more general result is now a consequence of Theorem 3.6:

Corollary 3.7. *Let B be a p -block of a finite group G with multiplicity one. Suppose that the fusion system \mathcal{F} of B is nonexotic (e.g. B is the principal block or G is p -solvable). Then the defect groups of B are elementary abelian.*

4. BLOCKS WITH FEW CHARACTERS

To finish this paper, we now concentrate on the subject of blocks with few characters. In this context, blocks B with $k(B) - l(B) = 1$ appear quite naturally, as we shall see. First of all, let us summarize a few facts on a p -block B of a finite group G with defect group D having a small number of characters. We have the following:

- (i) We have $k(B) = 1$ if and only if $D = 1$.
- (ii) We have $k(B) = 2$ if and only if $|D| = 2$ (see [2]).
- (iii) If $k(B) \leq 4$ and $l(B) = 1$, then $|D| = k(B)$ (see [17]).
- (iv) If $k(B) = 5$ and $l(B) = 1$, then $D \in \{C_5, D_8, Q_8\}$ (see [3]).

In this paper we conjecture that if $k(B) = 3$ then $|D| = 3$, and we prove that this is a consequence of the Alperin-McKay conjecture. First, we take care of the case where the defect group is normal.

Theorem 4.1. *Let B be a p -block of a finite group G with normal defect group D , and suppose that $k(B) = 3$. Then $|D| = 3$.*

Proof. By results of Fong and Reynolds, we may assume that D is a Sylow p -subgroup of G , and that $Z := \mathbf{O}_{p'}(G)$ is cyclic and central in G . By [17], we may also assume that $l(B) = 2$. Then G acts transitively on $D \setminus \{1\}$ by conjugation; in particular, D is elementary abelian. We write $|D| = p^d$. By the Hall-Higman Lemma, the kernel of the action of G on D is ZD . By a result of Passman [26], apart from finitely many exceptions, G/ZD is isomorphic to a subgroup of the semilinear group $T(p^d)$. Here $T(p^d)$ denotes the semidirect product of the multiplicative group $\mathbb{F}_{p^d}^\times$ of the finite field \mathbb{F}_{p^d} with the Galois group Γ of \mathbb{F}_{p^d} over \mathbb{F}_p . In particular, G/ZD has a cyclic normal subgroup H/ZD whose order s divides $p^d - 1$ such that G/H is

cyclic of order t dividing d . Since G/ZD acts transitively on $D \setminus \{1\}$ we also have $(p^d - 1) \mid |G : ZD| = st$.

It is well-known that $\text{IBr}(B) = \text{IBr}(G|\zeta)$ for some $\zeta \in \text{IBr}(Z)$. Let us consider $\text{IBr}(H|\zeta)$. On the one hand, $|\text{IBr}(G|\zeta)| = |\text{IBr}(B)| = l(B) = 2$ implies that G has at most two orbits on $\text{IBr}(H|\zeta)$. Moreover, each of these orbits has length at most $|G : H| = t$. Thus $|\text{IBr}(H|\zeta)| \leq 2t \leq 2d$.

On the other hand, we have $ZD/D \leq \mathbf{Z}(H/D)$. Since H/ZD is cyclic, H/D has to be abelian; in particular, we have $|\text{IBr}(H|\zeta)| = |H : ZD| = s$. Thus $s = |\text{IBr}(H|\zeta)| \leq 2d$, and $p^d - 1 \leq |G : ZD| \leq st \leq 2d^2$.

If $p = 2$, then our result follows easily since $k_0(B) \equiv 0 \pmod{4}$ for $d \geq 2$. Thus we may assume that $p \geq 3$.

If $d = 1$ then our result follows easily from the Brauer-Dade theory of blocks with cyclic defect groups. Thus we may assume that $d \geq 2$ and $p \geq 3$.

If $d = 2$ then $p^2 \leq 1 + 8 = 9$, i.e. $p = 3$. This case leads to a contradiction by making use of the results in [15]. Thus we may assume that $d \geq 3$ and $p \geq 3$, so that $3^d \leq p^d \leq 1 + d^2$; however, this is impossible.

It remains to deal with the exceptional cases in Passman's Theorem; so we may assume that

$$|D| \in \{3^2, 5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2, 3^4\}.$$

Suppose first that $d = 2$, and choose a nontrivial B -subsection (u, b_u) . Then b_u dominates a unique block \bar{b}_u of $\mathbf{C}_G(u)/\langle u \rangle$, and \bar{b}_u has defect 1. Since $1 = l(b_u) = l(\bar{b}_u)$ we conclude that \bar{b}_u has inertial index 1. Thus b_u has inertial index 1 as well, and G/ZD acts regularly on $D \setminus \{1\}$. Hence G/Z is a Frobenius group with Frobenius kernel ZD/Z and Frobenius complement G/ZD . In particular, the Sylow subgroups of G/ZD are cyclic or (generalized) quaternion. Thus the Schur multiplier of G/ZD is trivial. Hence we may assume that $Z = 1$. But then B is the only p -block of G , so that G has class number 3. This implies that $|G| \leq 6$, a contradiction.

We are left with the case $|D| = 3^4$. In this case G/Z is a doubly transitive permutation group of degree 3^4 . It is well-known that $|G/Z| = 2^k 3^4 5$ with $k \in \{5, 6, 7\}$ (see Example XII.7.4 in [12]). Using GAP [8], it is easy to work out the structure of G/Z and to derive a contradiction. \square

Theorem 4.2. *Let B be a p -block of a finite group G with defect group D such that $k(B) = 3$. Moreover, suppose that $k_0(B) = k_0(b)$ where b denotes the Brauer correspondent of B in $N_G(D)$. Then $|D| = 3$.*

Here $k_0(B)$ denotes the number of irreducible ordinary characters of height zero associated to B . Thus the conclusion of this theorem is satisfied if the Alperin-McKay Conjecture holds.

Proof of Theorem 4.2. It is well-known that b has defect group D . Then b dominates a unique p -block \bar{b} of $N_G(D)/\Phi(D)$, and \bar{b} has defect group $\bar{D} := D/\Phi(D)$ which is

abelian and normal in $N_G(D)/\Phi(D)$. Moreover, we have

$$k(\bar{b}) = k_0(\bar{b}) \leq k_0(b) = k_0(B) \leq k(B) = 3.$$

If we assume that $k(\bar{b}) \leq 2$, then we get $|\bar{D}| \leq 2$. Thus D is a cyclic 2-group which is impossible. This shows that we must have $k(\bar{b}) = 3$.

Since \bar{D} is normal in $N_G(D)/\Phi(D)$, Theorem 4.1 implies that $|\bar{D}| = 3$. Thus D is cyclic, and the Brauer-Dade theory yields the result. \square

It is known that $k_0(B) \geq 2$ for every block B with positive defect (see [24]). Hence, one may ask what can be said in case $k_0(B) = 2$. A similar proof as above shows that the Alperin-McKay Conjecture implies the following generalization of Brandt's result:

$$k_0(B) = 2 \iff k(B) = 2 \iff |D| = 2.$$

Finally, in this context of blocks with few characters, it might be of interest to recall that Héthelyi and Külshammer conjectured in [10] that

$$k(B) \geq 2\sqrt{p-1}$$

for arbitrary p -blocks B of arbitrary finite groups.

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REFERENCES

- [1] M. Aschbacher, R. Kessar and B. Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, Vol. 391, Cambridge University Press, Cambridge, 2011.
- [2] J. Brandt, *A lower bound for the number of irreducible characters in a block*, J. Algebra **74** (1982), 509–515.
- [3] M. Chlebowitz and B. Külshammer, *Symmetric local algebras with 5-dimensional center*, Trans. Amer. Math. Soc. **329** (1992), 715–731.
- [4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.
- [5] D. A. Craven, *Control of fusion and solubility in fusion systems*, J. Algebra **323** (2010), 2429–2448.
- [6] D. A. Craven, *The theory of fusion systems*, Cambridge Studies in Advanced Mathematics, Vol. 131, Cambridge University Press, Cambridge, 2011.
- [7] M. Fujii, *On determinants of Cartan matrices of p -blocks*, Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), 401–403.
- [8] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.5.7*; 2012, (<http://www.gap-system.org>).
- [9] D. Gorenstein and R. Lyons, *The local structure of finite groups of characteristic 2 type*, Mem. Amer. Math. Soc. **42** (1983).

- [10] L. Héthelyi and B. Külshammer, *On the number of conjugacy classes of a finite solvable group*, Bull. London Math. Soc. **32** (2000), 668–672.
- [11] L. Héthelyi, B. Külshammer and B. Sambale, *A note on Olsson’s Conjecture*, J. Algebra (to appear), DOI: 10.1016/j.jalgebra.2012.08.010.
- [12] B. Huppert and N. Blackburn, *Finite groups. III*, Grundlehren der Mathematischen Wissenschaften, Band 243, Springer-Verlag, Berlin, 1982.
- [13] R. Kessar and M. Linckelmann, *On blocks with Frobenius inertial quotient*, J. Algebra **249** (2002), 127–146.
- [14] R. Kessar and R. Stancu, *A reduction theorem for fusion systems of blocks*, J. Algebra **319** (2008), 806–823.
- [15] M. Kiyota, *On 3-blocks with an elementary abelian defect group of order 9*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), 33–58.
- [16] P. B. Kleidman and M. W. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, London Math. Soc. Lecture Note Ser. no. 129, Cambridge University Press, 1990.
- [17] B. Külshammer, *Symmetric local algebras and small blocks of finite groups*, J. Algebra **88** (1984), 190–195.
- [18] M. Liebeck, J. Saxl, and G. Seitz, *Subgroups of maximal rank in finite exceptional groups of Lie type*, Proc. London Math. Soc. **65** (1992), 297–325.
- [19] M. W. Liebeck and G. M. Seitz, *Unipotent and Nilpotent Classes in Simple Algebraic Groups and Lie Algebras*, Math. Surveys and Monographs Vol. 180, Amer. Math. Soc., Providence, 2012.
- [20] F. Lübeck, *Elements of order 2 and 3 in exceptional groups of Lie type*, <http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/23elts.html>
- [21] G. O. Michler, *On blocks with multiplicity one*, in: Representations of algebras (Puebla, 1980), 242–256, Lecture Notes in Math., Vol. 903, Springer, Berlin, 1981.
- [22] R. Narasaki and K. Uno, *Isometries and extra special Sylow groups of order p^3* , J. Algebra **322** (2009), 2027–2068.
- [23] G. Navarro and P. H. Tiep, *Abelian Sylow subgroups in a finite group*, preprint.
- [24] J. B. Olsson, *Inequalities for block-theoretic invariants*, in: Representations of algebras (Puebla, 1980), 270–284, Lecture Notes in Math., Vol. 903, Springer-Verlag, Berlin, 1981.
- [25] J. B. Olsson, *On subpairs and modular representation theory*, J. Algebra **76** (1982), 261–279.
- [26] D. S. Passman, *p -solvable doubly transitive permutation groups*, Pacific J. Math. **26** (1968), 555–577.
- [27] A. Ruiz and A. Viruel, *The classification of p -local finite groups over the extraspecial group of order p^3 and exponent p* , Math. Z. **248** (2004), 45–65.
- [28] Y. Tsushima, *A note on Cartan integers for p -solvable groups*, Osaka J. Math. **20** (1983), 675–679.

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