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Finite Groups with the Same Commuting Probability

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Abstract: Let G be a finite group. The commutativity degree of G, denoted by $Pr(G)$, is the probability that a randomly selected pairs of elements of the group commute. In this study we give an explicit formula for the number of conjugacy classes and commutativity degree of some finite non-abelian groups. In particular, we describe the commutativity degree of these groups both in the split and non-split case.

Keywords: Class equation, commutativity degree, conjugacy class

INTRODUCTION

The concept of commutativity degree or probability of commuting pairs of a group was established by Erdos and Turan (1968) and Gustafson (1973) and also studied by some researchers in different contexts such as Doostie and Maghasedi (2008), Erfanian and Russo (2008, 2009) and Lescot (1995). They have achieved to significant results on the lower and upper bound of the probability of commuting pairs of the finite groups. We should begin our investigation with a brief introduction and formal notations to commutativity degree. For a given finite group G of order n, the probability $Pr(G)$ that two elements selected at random from G are commutative is $\frac{|\Phi|}{n^2}$ where,

$$\Phi = \{(a,b) \in G^2 : [a,b] = 1\}$$

In order to count the elements of Φ, we have for each $a \in G$ the number of elements of Φ of the form (a,b) is $|C_G(a)|$, where $C_G(a)$ is the centralizer of a in G. Hence we have $|\Phi| = \sum_{a\in G} |C_G(a)|$. It follows that:

$$Pr(G) = \frac{1}{|G^2|}\sum_{a\in G}|C_G(a)| = \frac{1}{|G^2|}\sum_{i=1}^{k(G)}|a_i{}^G||C_G(a_i)| = \frac{1}{|G^2|}\sum_{i=1}^{k(G)}|G:C_G(a_i)||C_G(a_i)| = \frac{1}{|G^2|}\sum_{i=1}^{k(G)}|G| = \frac{k(G)|G|}{|G^2|}$$

Therefore,

$$Pr(G) = \frac{|\Phi|}{|G\times G|} = \frac{k(G)|G|}{|G|^2} = \frac{k(G)}{|G|} \quad (1)$$

where, $k(G)$ = the number of conjugacy classes of G?

One can refer to Gustafson (1973), Nath and Das (2010), Doostie and Maghasedi (2008) and Erdos and Turan (1968) for more details. The computational results on $Pr(G)$ are mainly due to Gustafson (1973) who shows that $\frac{5}{8}$ = the upper bound for $\frac{k(G)}{|G|}$, where G = a finite non-abelian group, thus $Pr(G) \le \frac{5}{8}$. The groups studied by Lescot (1995) mainly satisfy $\frac{1}{2} \le Pr(G) \le \frac{5}{8}$ and the obtained results of Doostie and Maghasedi (2008) concern the certain group's shows the property $Pr(G) < \frac{1}{2}$. Recently, Moradipour *et al.* (2012) showed that the precise value of $Pr(G)$ for a 2 generator metacyclic p-group $G = \frac{1}{p^\gamma} + \frac{1}{p^{\gamma+1}} - \frac{1}{p^{2\gamma+1}}$, for some integer γ.

Equation (1) shows that finding the commutativity degree of a finite group is equivalent to finding the number of conjugacy classes of the group. There are several papers on the conjugacy classes of finite p-groups including (Huppert, 1998; Sherman, 1979). For instance, in Sherman (1979) proved that if G is a finite nilpotent group of nilpotency class m, then $k(G) > m|G|^{\frac{1}{m}} - m + 1$ and then $k(G) > \log_2|G|$. Later Huppert (1998) showed that $k(G) > \log n$ for any nilpotent group G of order n. In Moradipour *et al.* (2012) showed that the exact number of conjugacy classes of metacyclic p-groups are $p^{\alpha+\beta}(\frac{1}{p^\gamma} + \frac{1}{p^{\gamma+1}} - \frac{1}{p^{2\gamma+1}})$, for some integers α, β, γ.

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The class equation in a finite group G is often written in terms of the center, centralizer and the number of conjugacy classes. It can be related to the commutativity degree of the group. To write the class equation of G, if $\{[x_i]: 1 \leq i \leq k(G)\}$ is the set of distinct conjugacy class of the group then the class equation is in the form:

$$|G| = |Z(G)| + \sum_{i=|Z(G)|+1}^{k(G)} |G: C_G(x_i)| \\ = |Z(G)| + \sum_{i=|Z(G)|+1}^{k(G)} |x_i{}^G| \qquad (2)$$

Regarding the above equation, to compute the commutativity degree of a group, it is often easier to find the size of each centralizer $C_G(x_i)$ in Eq. (2) than to compute the number of conjugacy classes.

In this study we compute the commutativity degree of the generalized quaternion group $Q_{2^{m+1}}$, dihedral group $D_{2^{m+1}}$ and semi-dihedral group $SD_{2^{m+1}}$ by using the class equations. In particular, we will show that these groups in the split and non-split case have the same conjugacy classes and commutativity degree.

PRELIMINARIES

In this section we give some results which are needed to prove our main results. If $a, b \in G$ then the notation $[b, a] = bab^{-1}a^{-1} = a^b a^{-1}$ is the commutator of b and a. We denote by $k(G)$, $Pr(G)$ the number of conjugacy classes and commutativity degree of group G, respectively.

Lemma 1: Let m, n, r, s and t be integers with m, n non-negative and let $G(2, m, n, s, t) \simeq \langle a, b | a^{2^m} = 1, b^{2^n} = a^{2^{m-s}}, a^b = a^\mu \rangle$, be a non-abelian 2-group, where $\mu = 2^{m-t} - 1$. If $x, y \in G$ with $x = a^\sigma b^\lambda$ and $y = a^k b^l$ then the following hold in G:

- $b^\lambda a^\sigma = a^{\sigma\mu^\lambda} b^\lambda$
- $xy = a^{\sigma + k\mu^\lambda} b^{\lambda + l}$
- $x^y = a^{k(1-\mu^\lambda) + \sigma\mu^l} b^\lambda$
- $[x, y] = a^{\sigma(1-\mu^l) + k(\mu^\lambda - 1)}$

Proof: Using induction on λ, it is easy to see that $a^{b^\lambda} = a^{\mu^\lambda}$ for each $\lambda \geq 0$. In fact $a^b = a^\mu$ and if $a^{b^\lambda} = a^{\mu^\lambda}$, then $a^{b^{\lambda+1}} = \left(a^{b^j}\right)^b = \left(a^{\mu^\lambda}\right)^b = (a^\mu)^{\mu^\lambda} = a^{\mu^{\lambda+1}}$. Thus $a^{\sigma b^\lambda} = a^{\sigma\mu^\lambda}$ that is $b^\lambda a^\sigma b^{-\lambda} = a^{\sigma\mu^\lambda}$. Hence $b^\lambda a^\sigma = a^{\sigma\mu^\lambda} b^\lambda$ and the result follows: ■

The presentation $G(2, m, n, s, t)$ in Lemma 1 is a met acyclic 2-group which is the extension of a cyclic normal subgroup $\langle a \rangle$ by $\langle b \rangle$. The following lemma gives the order, center and the order of the center of a metacyclic 2-group.

Lemma 2: Let G be a metacyclic 2-group of type $G(2; m, n, s, t)$, where $m, n \in \mathbb{N}, m \geq 2t, n \geq t \geq 1$ and $\mu = 2^{m-t} - 1$. Then:

- $|G| = 2^{m+n}$
- $Z(G) = \langle a^{2^{m-1}}, b^{2^{\max\{1,\gamma\}}} \rangle$
- $|Z(G)| = 2^{n - max\{1,\gamma\}+1}$

Proof: For (1) $G = \langle a, b \rangle = \langle a \rangle \langle b \rangle$, $|a| = 2^m$ and $|b| = 2^{n+s}$. Also, $\langle a \rangle \cap \langle b \rangle = \langle b^{2^n} \rangle = \langle a^{2^{m-s}} \rangle$ and $(b^{2^n})^{2^s} = (a^{2^{m-s}})^{2^s} = 1$. Thus the order of G is:

$$|G| = \frac{|\langle a \rangle||\langle b \rangle|}{|\langle a \rangle \cap \langle b \rangle|} = \frac{2^m \cdot 2^{n+s}}{2^s} = 2^{m+n}$$

For parts (2) and (3) we refer to King (1973) Proposition 4.10. ■

The following corollary shows when a representation of a met acyclic 2-group is a split. For the proof, we refer to the above lemma and Beuerle (2005).

Corollary 1: Let G be a group of type $\mathrm{G}(2; \mathrm{m}, \mathrm{n}, 1, \mathrm{t})$, where $\mathrm{t} = 2^{\mathrm{m-r}} - 1$. If $\mathrm{n} = \mathrm{t}$, then G is isomorphic to a split metacyclic 2-group and in particular, $\mathrm{G} \simeq \mathrm{G}(2;\ \mathrm{m}, \mathrm{n}, 0, \mathrm{n})$.

According to King (1973) and Beuerle (2005) a met acyclic group either splits or never splits. Correspondingly, its uniquely reduced presentation $G(2; m, n, s, t)$ has either $s = 0$ (splitting) or $s > 0$ (non-split).

Now we define the following 3 finite groups which can be shown by reduced presentation $G(2; m, 1, s, t)$, where, $0 \leq s, t \leq 1$:

- $G(2; m, 1,1,0) = \langle a, b | a^{2^m} = 1, b^2 = a^{2^{m-1}}, a^b = a{-}1 \simeq Q2m{+}1$
- $G(2; m, 1,0,0) = \langle a, b | a^{2^m} = b^2 = 1, a^b = a{-}1 \simeq D2m{+}1$
- $G(2; m, 1,0,1) = \langle a, b | a^{2^m} = b^2 = 1, a^b = a2m{-}1{-}1 \simeq SD2m{+}1$

where, $Q_{2^{m+1}}$, $D_{2^{m+1}}$ and $SD_{2^{m+1}}$ are the generalized quaternion, dihedral and semidihedral groups, respectively.

MAIN RESULTS

Now we are ready to prove our main theorem. This theorem gives a formula for the number of conjugacy classes and commutativity degree of the generalized quaternion, dihedral and semi-dihedral groups in terms of m. On the other hand, for the group $G(2; m, n, s, t)$ with relation $ba = a^{\mu} b$, each element can be written in the unique form $a^i b^j$, where all possible values for i and j is $0 \leq i < 2^m, 0 \leq j < 2^n$.

Theorem 1 (main result): Let $G \simeq G(2;\, m, 1, s, t)$ be a generalized quaternion, dihedral or semidihedral group. Then:

- $k(G) = \begin{cases} 2^m + 1, & a^i b^j \notin Z(G) \text{ and } j \text{ even} \\ \frac{3 \cdot 2^{m-1} - 1}{2^{m-2}}, & a^i b^j \notin Z(G) \text{ and } j \text{ odd} \\ 2^{m+1}, & a^i b^j \in Z(G) \end{cases}$

- $Pr(G) = \begin{cases} \frac{1}{2} + \frac{1}{2^{m+1}}, & a^i b^j \notin Z(G) \text{ and } j \text{ even} \\ \frac{3 \cdot 2^{m-1} - 1}{2^{\,2m-1}}, & a^i b^j \notin Z(G) \text{ and } j \text{ odd} \\ 1, & a^i b^j \in Z(G) \end{cases}$

where, $a^i b^j \in G$ and a^i is a non-central element of G.

Proof: According to Lemma 2 we have $Z(G) = \langle a^{2^{m-1}}, b^2 \rangle$, with $|Z(G)| = 2$, since $\gamma = 1$. We also have $|G| = 2^{m+1}$. Using Corollary 1 the group (1) is non-split and the groups (2) and (3) are split. Recall that an arbitrary element of G can be written uniquely in the form $a^i b^j$ where, $0 \leq i < 2^m, 0 \leq j < 2^n$. Let $g = a^i b^j \in G$ and a^i be a non-central element of G. By using Lemma 1 it is easy to see that if j is even then $|C_G(a^i b^j)| = 2^m$. Also, using Lemma 1 and a simple verification we can see that if j is odd then $|C_G(a^i b^j)| = 4$. From the other point of view, using Eq. (2) yields:

$$|G| = |Z(G)| + \sum_{i=|Z(G)|+1}^{k(G)} \frac{|G|}{|C_G(x_i)|}$$

Thus we have:

$$2^{m+1} = 2 + \sum_{i=3}^{k(G)} \frac{2^{m+1}}{2^m} = 2 + (k(G) - 2)\left(\frac{2^{m+1}}{2^m}\right)$$

Hence we conclude that $2^{m+1} - 2 = 2(k(G) - 2)$. Hence, $k(G) = 2^m + 1$, if j is even. Similarly, using class Eq. (2) we have:

$$2^{m+1} = 2 + \sum_{i=3}^{k(G)} \frac{2^{m+1}}{4} = 2 + (k(G) - 2)(2^{m-1})$$

It follows that $2^{m+1} + 2^m - 2 = 2^{m-1} k(G)$. Hence $k(G) = \frac{3 \cdot 2^{m-1} - 1}{2^{m-2}}$, if j is odd. Therefore,

$$k(G) = \begin{cases} 2^m + 1, & a^i b^j \notin Z(G) \text{ and } j \text{ even} \\ \frac{3 \cdot 2^{m-1} - 1}{2^{m-2}}, & a^i b^j \notin Z(G) \text{ and } j \text{ odd} \\ 2^{m+1}, & a^i b^j \in Z(G) \end{cases} \quad (3)$$

Now using Eq. (1) yields:

$$Pr(G) = \begin{cases} \frac{1}{2} + \frac{1}{2^{m+1}}, & a^i b^j \notin Z(G) \text{ and } j \text{ even} \\ \frac{3 \cdot 2^{m-1} - 1}{2^{\,2m-1}}, & a^i b^j \notin Z(G) \text{ and } j \text{ odd} \\ 1, & a^i b^j \in Z(G) \end{cases} \quad (4)$$

as desired. ■

As an example, we give the group of quaternion of order 8 as follows.

Example 1: If $G \simeq Q_8 = \langle a, b : a^4 = 1, b^2 = [b, a] = a^{-2}\rangle$, the group of quaternion of order 8, then $Pr(G) = \frac{5}{8}$.

Proof: Using Eq. (4) with $m = 2$, $a^i b^j \notin Z(G)$ and j even the result follows. ■

CONCLUSION

In this study, we have shown that the commutativity degree of non-abelian generalized quaternion group $Q_{2^{m+1}}$, dihedral group $D_{2^{m+1}}$ and semi-dihedral group $SD_{2^{m+1}}$ are the number of conjugacy classes of these groups are the same.

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