

About the effects of polarising optics on lidar signals and the $\Delta 90$ -calibration

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Abstract

This paper provides a model for assessing the effects of polarising optics on the signals of typical lidar systems, which is based on the description of the individual optical elements of the lidar and of the state of polarisation of the light by means of the Müller-Stokes formalism. General analytical equations are derived for the dependence of the lidar signals on polarisation parameters, for the linear depolarisation ratio, and for the signals of different polarisation calibration setups. The equations can also be used for the calculation of systematic errors caused by non-ideal optical elements, their rotational misalignment, and by non-ideal laser polarisation. We present the description of the lidar signals including the polarisation calibration in a closed form, which can be applied for a large variety of lidar systems.

1 Introduction

The purpose of atmospheric depolarisation measurements with lidar, first described by Schotland et al. (1971), is mainly to discern between more or less depolarising scatterers. The discrimination of ice and water clouds was the main focus in the beginning. Sassen (1991) and Sassen (2005) give an overview about the early work related to that. Aerosol and their interaction with clouds became more important in the last decade because of their insufficiently understood direct and indirect roles in the feedback mechanisms of climate change (Boucher et al. 2013). Multi-wavelength lidar measurements including the depolarisation ratio can be used to discern aerosol types (Sugimoto et al. 2002; Sugimoto and Lee 2006; Ansmann et al. 2011; Burton et al. 2014; Groß et al. 2014) and to retrieve micro-physical aerosol properties by means of inversion algorithms (Müller et al. 1999; Ansmann and Müller 2005; Gasteiger et al. 2011; Veselovskii et al. 2013; Böckmann and Osterloh 2014; Müller et al. 2014). Pérez-Ramírez et al. (2013) show the impact of systematic errors of the lidar data on the retrieval of micro-physical particle properties. The additional measurement of the linear or circular depolarisation ratio improves the retrievals (Böckmann and Osterloh 2014; Gasteiger and Freudenthaler 2014). But the depolarisation ratios are often derived from lidar measurements assuming more or less ideal lidar setups neglecting the effects of small system misalignments and of non-ideal optical elements on the polarisation, which can lead to considerable errors in the retrieved depolarisation ratio (Reichardt et al. 2003; Alvarez et al. 2006; Freudenthaler et al. 2009; Mattis et al. 2009). According to Chipman (2009a) Chap. 15.27, one of the primary difficulties in performing accurate polarisation measurements is the systematic error due to non-ideal polarisation elements. Most inclined optical surfaces and optical coatings on beam-splitters are polarising, wherefore all lidars must be considered as

“incomplete light-measuring polarimeters” (Chipman 2009a), even if they are not intended to measure the depolarisation ratio.

As model calculations of aerosol scattering properties advance (Nousiainen et al. 2011; Kahnert et al. 2014), the modellers need accurate measurements with small errors and reliable error bars in order to verify and improve their models. In order to estimate the uncertainties and to improve the measurements, we have to find the error sources. The usual way to do this is to compare the measurements with a model and to investigate the deviations. The only reliable atmospheric model for comparison is the model of the molecular linear depolarisation ratio δ_m (Behrendt and Nakamura 2002; Freudenthaler et al. 2015). But the actually measured values δ_m^* of the very small real δ_m (on the order of 0.004) are usually a number of times higher, which makes it difficult to use them for the calibration with a simple model as $\delta_m^* = A\delta + B$ ((Sassen and Benson 2001; Reichardt et al. 2003) (see also S.9). At present, polarisation calibration techniques of lidars are often not accurate enough to sufficiently determine the two parameters A and B , and actually, as we will show in the following, the model itself is insufficient. But how accurate do we have to be? How accurate can we be? Which are the critical parts and adjustments? How can setups be improved with minimal costs and complexity, and how can existing lidar systems be checked? To answer these questions, we need a better model for the lidar setup, which is complete and flexible enough to be applied to a variety of lidar systems and can describe various calibration techniques.

Astronomical polarisation measurement setups are very similar to lidar setups. Elaborate theoretical and experimental investigations of the influence of polarising optics and corresponding corrections for astronomical telescopes and detection optics using the theory of polarimetry and ellipsometry (see Azzam (2009); Chipman (2009a)) can be found quite frequently in the literature (Skumanich et al. 1997; Socas-Navarro et al. 2011; Breckinridge et al. 2015). Although the usefulness of a lidar with polarisation diversity had been realised early (Pal and Carswell 1973), the need for a complete description with the Müller-Stokes formalism has, to our knowledge, been first expressed by Anderson (1989), but focused only on the atmospheric scattering process. Instrumental aspects including some error calculations have been included by Beyerle (1994), Cairo et al. (1999), Biele et al. (2000), Behrendt and Nakamura (2002), Reichardt et al. (2003), Alvarez et al. (2006), Del Guasta et al. (2006), Hayman and Thayer (2009), Mattis et al. (2009), Freudenthaler et al. (2009), Hayman (2011), Hayman and Thayer (2012), David et al. (2013), Geier and Arienti (2014), Di et al. (2015), and Volkov et al. (2015). The errors mainly considered are the diattenuation of the receiver optics (see Sect. 2.2), the cross talk of the polarising beam-splitter, non-ideal characteristics of the calibration, and rotational misalignment of polarising components.

In this work we describe lidar setups from the laser to the detector by means of the Stokes-Müller formalism (Chipman 2009b) including the transmitter and receiver optics. The Stokes vector describes the flux and the state of polarisation of the light, and the Müller matrices describe how optical elements change the Stokes vector. We develop equations for the two signals of a polarisation sensitive lidar and for the signals of the polarisation calibration, which are necessary to retrieve the linear depolarisation ratio and the total lidar signal, using different calibration techniques and lidar setups. In order to enable the evaluation of the final errors and to analyse their dependencies on certain optical parameters or misalignments of individual optical elements, we derive first the full equations and then try to find more simple analytical formulations neglecting minor error sources to get an overview of the main critical parameters. For this we neglect the polarisation effects of lenses and of telescope mirrors with small incidence angles of the light beam (Seldomridge et al. 2006; Clark and Breckinridge 2011). Although not considered here, 90° folding mirrors as in Newtonian-type telescopes

(Breckinridge et al. 2015; Di et al. 2015) and stress-birefringence in windows and lenses or unfavourable coatings might cause severe polarisation effects. This requires further investigation. In general, errors caused by a light beam which is divergent or inclined towards the optical axis are not discussed here; this means the light beams are assumed to be either perfectly parallel before and after polarisation optics, or that an optical element is insensitive to the incident angle regarding polarisation.

Basic information about the polarisation topics can be found in Goldstein (2003), Clarke (2009), and in the chapters by Azzam (2009); Bennett (2009a,b); Chipman (2009b,a) of the 3rd edition of the Handbook of Optics (Bass 2009). The authors of these chapters follow the Muller-Nebraska convention Muller (1969) for the definition of signs and directions regarding e.g. the coordinate system (see Suppl. S.1), as we do in this work.

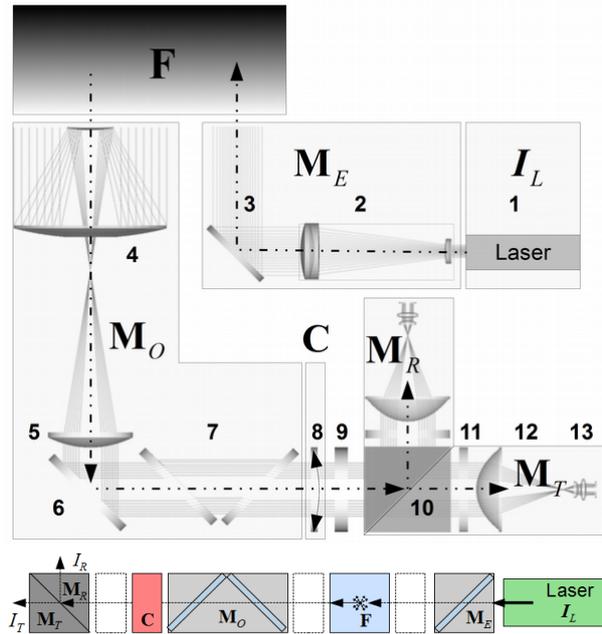


Figure 1 Top: Exemplary depolarisation lidar setup with laser 1, beam expander 2, steering mirror 3, receiving telescope 4, collimator 5, folding mirror 6, dichroic beam-splitters 7, a rotating element for polarisation calibration 8, interference filter 9, and polarising beam-splitter cube 10 (PBS, polarising beam-splitter). The neutral density filters and cleaning polarisers 11, detector optics 12, and the detectors 13. The system can be subdivided in functional blocks which can be described with the Stokes-Müller formalism: I_L is the Stokes vector of the laser source, M_E is the Müller matrix of the laser emitter optics, F of the atmospheric backscattering volume including depolarisation, M_O includes receiver optics as beam-splitters, C is the calibrator, and $M_{T,R}$ is the polarising beam-splitter including the detector optics for the transmitted (T) and reflected (R) optical branches. Bottom: simplified schematic of the setup.

Most of the lidar setups for depolarisation measurement reported in the literature are explicable with the schematic in Fig. (1), in which the individual parts of a lidar system are grouped in modules, which are in general describable by Müller matrices of combinations of diattenuators, retarders, and rotators (see Sect.2.2). The setup in Fig. 1 can be described with Eq. (1).

$$I_{T,R} = \eta_{T,R} \mathbf{M}_{T,R} \mathbf{C} \mathbf{M}_O \mathbf{F} \mathbf{M}_E I_L \quad (1)$$

Symbols for Müller matrices are bold (\mathbf{M}), vectors are bold-italic (\mathbf{I}), and variables italic (I). The laser beam with Stokes vector I_L is expanded and directed towards the atmosphere with backscatter matrix \mathbf{F} by the emitter module with Müller matrix \mathbf{M}_E . The backscattered photons are received by the telescope with a subsequent collimation lens and dichroic beam-

splitters in the receiver optics module \mathbf{M}_O . A polarisation calibrator with Müller matrix \mathbf{C} is placed here before the polarising beam-splitter cube (10) with Müller matrices \mathbf{M}_T for the transmitted and \mathbf{M}_R for the reflected path, their opto-electronic gains $\eta_{T,R}$, and the final Stokes vectors $\mathbf{I}_{T,R}$ at the detectors. The opto-electronic gains $\eta_{T,R}$ include the attenuation of all non-polarising optical elements as neutral density and bandpass filters, the efficiency of the detectors, and the amplification of the electronic system. The scattering volume \mathbf{F} can be at any distance from the lidar (lidar-range), because we assume that the extinction in the range between the lidar and the scattering volume \mathbf{F} is polarisation independent and that signal contributions due to forward or multiple scattering in this range can be neglected. Therefore we neglect all lidar-range dependencies in the following equations. We also do not consider range dependent effects as the overlap function and the range dependent transmission of interference filters and dichroic beam-splitters, which is caused by the range dependent incident angles on the optics.

Various lidar systems employ different calibration techniques with calibrating devices with Müller matrix \mathbf{C} at different places in the optical setup, with the respective equations:

$$\text{before the polarising beam-splitter} \quad \mathbf{I}_S = \eta_S \mathbf{M}_S \mathbf{C} \mathbf{M}_O \mathbf{F} \mathbf{M}_E \mathbf{I}_L \quad (2)$$

$$\text{before the receiver optics} \quad \mathbf{I}_S = \eta_S \mathbf{M}_S \mathbf{M}_O \mathbf{C} \mathbf{F} \mathbf{M}_E \mathbf{I}_L \quad (3)$$

$$\text{behind the laser emitter optics} \quad \mathbf{I}_S = \eta_S \mathbf{M}_S \mathbf{M}_O \mathbf{F} \mathbf{C} \mathbf{M}_E \mathbf{I}_L \quad (4)$$

$$\text{before the laser emitter optics} \quad \mathbf{I}_S = \eta_S \mathbf{M}_S \mathbf{M}_O \mathbf{F} \mathbf{M}_E \mathbf{C} \mathbf{I}_L \quad (5)$$

In the following we list just a few examples from the literature with sufficient description of their calibration technique. Pal and Carswell (1973) used three telescopes with Glan-Thomson prisms in the receiver optics (Eq. (2)) at 0° , 45° , and 90° orientation with respect to the laser polarisation to determine the first three Stokes parameters of the scattered light, and calibrated them by mechanically switching all polarisers to 0° orientation. Houston and Carswell (1978) extended this setup by a fourth telescope with a $\lambda/4$ plate to measure all four Stokes parameters, with the same calibration technique as before. The relative polarisation sensitivity of the CALIOP lidar on CALIPSO (Winker et al. 2009) is calibrated with a pseudo-depolariser before the polarising beam-splitter (Hunt et al. 2009), which is described by Eq. (2). Del Guasta et al. (2006) calibrate the gain ratio η_R/η_T of their polarimetric lidar with an unpolarised light source before the polarising beam-splitter (Eq. (2)) and determine the receiving optics Müller matrix \mathbf{M}_O with a linearly polarised light source and rotating the receiving optics, which corresponds to Eq. (3) with a mechanical rotation matrix \mathbf{C} . Similar rotation calibration before the polarising beam-splitter is applied with RALI (Nemuc et al. 2013) and the Raymetrics LR331D400 (Bravo-Aranda et al. 2013) with a mechanical rotation $\Delta 90^\circ$ -calibration (see Sect. 5), and with a $\lambda/2$ plate rotation in the MULIS (Freudenthaler et al. 2009) and the Cloud Physics Lidar (McGill et al. 2002; Liu et al. 2004). A sheet polariser at 45° is used before the polarising beam-splitter in the AD-Net lidars (Shimizu et al. 2004). Mechanical rotation before the receiving optics (Eq. (3)) is employed for the DLR HSRL (Esselborn et al. 2008), for POLIS (Freudenthaler et al. 2009), and by Nisantzi et al. (2014). For the McMurdo lidar (Snels et al. 2009) and the PollyXT (Engelmann et al. 2015) a linear polariser is used before the receiving optics. An unpolarised light source before the receiver telescope is used by Mattis et al. (2009). Spinhirne et al. (1982) use a $\lambda/2$ plate for polarisation rotation in the output beam (Eq. (4)). The HSRL-1 (Hair et al. 2008) and HSRL-2 (Burton et al. 2015) as well as David et al. (2012) use a $\lambda/2$ plate as rotation calibrator before some parts of the emission optics (Eq. (5)). Roy et al. (2011) and Cao et al. (2010) use a $\lambda/2$

plate before the emitter optics (Eq. (5)), but they switch the plane of emitted polarisation continually between horizontal and vertical and calculate the linear depolarisation ratio from the geometric mean of both measurements, which makes a separate calibration unnecessary. However, the equations of this work can still be used for the error analysis. Polarisation switching between laser pulses and with only one detection channel is done by Platt (1977) with mechanical rotation of the receiver optics, by Eloranta and Piironen (1994) with a $\lambda/2$ plate after the emitter optics (Eq. (4)), by Seldomridge et al. (2006) with a nematic liquid crystal before the polarising beam-splitter (Eq. (2)), and by (Flynn et al. 2007) with a $\lambda/2$ plate before the emitter optics (Eq. (5)). Although the explicit equations in this work consider only one variable polarising element (i.e. the calibrator), the equations for more complex lidar setups as with a polarising beam-splitter and a $\lambda/4$ plate in the common emitter/receiver path ((Eloranta 2005; David et al. 2013) or with different variable polarisation elements in the emitter/receiver path (Kaul et al. 2004; Hayman et al. 2012; Volkov et al. 2015) can be constructed with the equations provided in this work. Snels et al. (2009) present an overview of some potential error sources and other existing polarisation calibration techniques including calibration with assumed known depolarisation from molecules (“clear sky”) or clouds with spherical particles.

The equations presented in this work can be used for the design of lidar systems, especially for the determination of the requirements for certain components in order to achieve the desired measurement accuracy, for the analysis of the performance of existing lidar systems by means of different calibration setups, and for the final error calculation with respect to the polarisation characteristics.

One of the main uncertainties is the orientation of the plane of polarisation of the laser beam (angle α) with respect to the orientation of the polarising beam-splitter (briefly laser rotation), because first, the plane of polarisation of the laser might not only be determined by the orientation of the Pockels cell in the laser cavity, but also by the orientation of the crystals for second and third harmonics generation and by the harmonic separation beam-splitters. Second, the laser and emitter optics are often mounted on a separate optical breadboard, which might be rotated with respect to the receiver breadboard. Furthermore, laser manufacturers usually provide neither an indication of the accuracy of the orientation nor an accurate mechanical reference for it, the orientation cannot be measured easily, and finally, the orientation can change with time and environmental conditions. We take into account that in lidar labs it is usually not possible to perform elaborate and accurate measurements as in an optical lab equipped for ellipsometric measurements. Therefore we want to use simple tools and as few as possible measurements - at best with the tools which we already use for the standard depolarisation measurements.

Some optical parts can be made almost ideal and some misalignments can be made very small so that they become negligible. For these cases often much simpler equations can be derived, which show the residual influence of the other non-ideal parts, and which can be used directly in lidar retrieval algorithms. It becomes also clear in which cases corrections are not possible, when additional measurements with simple setups can help to retrieve the properties of the disturbing parts, and where one has to be careful in the design of a lidar system to avoid non-correctable errors. We want to find the setups and calibrators, with which the calibration can be measured with the least errors, and we want equations to assess the final uncertainties in the retrieved lidar products. Setups with 90° separated limit stops can be made very accurate ($< 0.1^\circ$) by means of working machines. Motorised holders with sufficient resolution and accuracy are commercially available. An example for an almost ideal part is the linear polariser. Polarising sheet filters are available with high extinction, well specified by

manufacturers. They are relatively insensitive to the incident angle, work over a sufficiently large wavelength range, and are thin, which means that they can be placed even in already existing lidar systems with little space for additional optics. Additionally, they are available in large size at an affordable price - in contrast to crystal polarisers and wave plates, and thus they can also be placed before the telescope. Wave plates and circular polarisers made of plastic sheets are usually not as well specified concerning their phase shift, acceptance angle and wavelength range. For other places, which require only small diameters, true zero-order $\lambda/2$ plates can be used.

Since the atmosphere is not stable and the laser power might change between two consecutive measurements, the absolute signals change. But if we use the ratios of the cross and parallel signals, which only change with the atmospheric polarisation parameter a , we can easily find atmospheric situations which introduce negligible errors in the calculations. Therefore we only use signal ratios for the calibrations.

Most of the problems can probably be solved with a much smaller theoretical framework. But then often questions arise, how the one or other misalignment, rotation, additional retardance or diattenuation would influence the final results. The impotence of less extended formulations to answer these questions will always leave an uncomfortable uncertainty. This work is an attempt to provide the tools to answer some of these questions, with the disadvantage of being rather extended.

Section 2 provides a simplified example as an introduction and preparation for Sect. 3, where we introduce the concepts and parameters which are necessary to formulate the equations in such a general way that they can be applied to a large variety of lidar systems. In order to generalise and to simplify the expressions, several binary parameters are introduced in the equations, which enable us to describe orthogonal orientations of individual elements with just one expression and which reduce the number of equations considerably. In Sect. 4 we develop the general equations for the lidar signals of normal atmospheric measurements (*standard measurements* in the following) and for the linear depolarisation ratio. In Sect. 5 we introduce the general concept of the 45° and $\Delta 90$ calibrations, which is then applied in Sect. 6 to 10 for different calibrators and in the subsections for different positions of the calibrators in the emitter-receiver optics. We include the following types of calibrators: unpolarised light (Sect. 6), which has to be inserted by an additional light source or diffuser and has therefore some disadvantages; the mechanical and $\lambda/2$ plate rotator (Sect. 7); the linear polariser (Sect. 8), which can be easily included in existing systems; the $\lambda/4$ plate (Sect. 9), which can also be used to determine the amount of circular polarisation, and the circular polariser (Sect. 10). General purpose equations used in several sections are shifted to the appendices, and common equations or concepts, which can also be found in standard text books, are collected in the supplement in order to show their form with the variables used in this work.

2 The basic Müller-Stokes representation of lidar signals with polarisation

In this chapter we use a simple example of Fig.(1), described with Eq. (2), to introduce some basic concepts. It contains a calibrator \mathbf{C} before the polarising beam-splitter and neglects the polarising effects of the receiver optics \mathbf{M}_O , i.e.

$$\mathbf{I}_{T,R} = \eta_{T,R} \mathbf{M}_{T,R} \mathbf{C} \mathbf{F} \mathbf{I}_L \quad (6)$$

The total power I_L and the state of polarisation of horizontal-linear polarised laser light are represented by the Stokes vector

$$\mathbf{I}_L = I_L \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad (7)$$

The magnitude I_L of the Stokes vector is the total light beam intensity. It is directly measurable with a light detector for the flux of photons. Because a lidar includes optics as telescope and lenses, which change the diameter or focus the light beam, here the colloquial intensity means the radiant flux or radiant energy per unit time. However, the finally measured quantities are the electronic signals I_T and I_R of the detectors in the transmitted and reflected paths. We use flux, intensity and signal alternatively, depending on the context.

2.1 Depolarising atmospheric aerosol

Müller matrices describe the linear interaction between polarised light and an optical system (optical elements or medium). For any input, represented as a Stokes vector, the Müller matrix produces a unique output, in the form of another Stokes vector. For the backscattering of a volume of randomly oriented, non-spherical particles with rotation and reflection symmetry the Müller matrix \mathbf{F} can be written as (van de Hulst 1981; Mishchenko and Hovenier 1995; Mishchenko et al. 2002)

$$\mathbf{F} = \begin{pmatrix} F_{11} & 0 & 0 & 0 \\ 0 & F_{22} & 0 & 0 \\ 0 & 0 & -F_{22} & 0 \\ 0 & 0 & 0 & F_{44} \end{pmatrix} = F_{11} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1-2a \end{pmatrix} \quad (8)$$

with the polarisation parameter a (Chipman 2009b; Eq. (93))

$$a = \frac{F_{22}}{F_{11}} \quad (9)$$

and

$$F_{44} = F_{11} - 2F_{22} = F_{11}(1 - 2a) \quad (10)$$

Note, that in some literature (Flynn et al. 2007; Gimmestad 2008; Roy et al. 2011; Gasteiger and Freudenthaler 2014) the *de-polarisation parameter* $d = (1 - a)$ is used, and in Borovoi et al. (2014) d is called *polarisation parameter*. In Volkov et al. (2015) $e = a$ (for randomly oriented particles) is called *sphericity index*. However, in this work we use the polarisation parameter a for the reason of brevity, which is the fraction of the backscattered light that maintains the emitted linear polarisation.

The matrix \mathbf{F} in Eq. (8) describes a pure depolariser \mathbf{M}_Δ (Lu and Chipman 1996), but including a mirror reflection \mathbf{M}_M for the backscattering direction, with the backscatter coefficient F_{11} .

$$\mathbf{F} = \mathbf{M}_M \mathbf{M}_\Delta = F_{11} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 2a-1 \end{pmatrix} \quad (11)$$

F_{11} and a are the only range dependent parameters in all the following equations. The volume linear depolarisation ratio δ of the scattering volume, which contains particles and air molecules, can be written as (Mishchenko and Hovenier 1995)

$$\delta = \frac{F_{11} - F_{22}}{F_{11} + F_{22}} = \frac{1 - a}{1 + a} \Rightarrow a = \frac{1 - \delta}{1 + \delta} \quad (12)$$

The Stokes vector \mathbf{I}_{in} of horizontal-linear polarised light I_L reflected by the atmosphere \mathbf{F} and incident in the receiving optics is

$$\mathbf{I}_{in} = \mathbf{F}\mathbf{I}_L = F_{11} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1 - 2a \end{pmatrix} \mathbf{I}_L = F_{11} \mathbf{I}_L \begin{pmatrix} 1 \\ a \\ 0 \\ 0 \end{pmatrix} \quad (13)$$

2.2 Optical parts: diattenuator with retardation

All other optical elements in the lidar receiver can be described as a combination of diattenuators and retarders (Lu and Chipman, 1996) (retarding diattenuators; Eq. (14)). Often a polarising beam-splitter cube is used for splitting in transmitted and a reflected components polarised parallel and perpendicular with respect to the laser polarisation. But also polarising or even non-polarising beam-splitter plates with subsequent polarisation filters (analysers) can be used. All of them and combinations of them can be described with the Müller matrix of a polarising beam-splitter (PBS) (Pezzaniti and Chipman 1994), considering the remarks in S.4. The matrix of the transmitting part is

$$\mathbf{M}_T = \frac{1}{2} \begin{pmatrix} T_T^p + T_T^s & T_T^p - T_T^s & 0 & 0 \\ T_T^p - T_T^s & T_T^p + T_T^s & 0 & 0 \\ 0 & 0 & 2\sqrt{T_T^p T_T^s} \cos \Delta_T & 2\sqrt{T_T^p T_T^s} \sin \Delta_T \\ 0 & 0 & -2\sqrt{T_T^p T_T^s} \sin \Delta_T & 2\sqrt{T_T^p T_T^s} \cos \Delta_T \end{pmatrix} = \quad (14)$$

$$= T_T \begin{pmatrix} 1 & D_T & 0 & 0 \\ D_T & 1 & 0 & 0 \\ 0 & 0 & Z_T c_T & Z_T s_T \\ 0 & 0 & -Z_T s_T & Z_T c_T \end{pmatrix}$$

with the intensity transmission coefficients (transmittance) for light polarised parallel (T^p) and perpendicular (T^s) to the plane of incidence of the PBS, the diattenuation parameter D_T , and the average transmittance T_T , i.e. for unpolarised light. Δ_T is the difference of the phase shifts of the parallel and perpendicular polarised electrical fields (retardance) according to the Muller-Nebraska convention (Muller 1969).

$$T_T = \frac{T_T^p + T_T^s}{2}, \quad D_T = \frac{T_T^p - T_T^s}{T_T^p + T_T^s}, \quad Z_T = \frac{2\sqrt{T_T^p T_T^s}}{T_T^p + T_T^s} = \sqrt{1 - D_T^2}, \quad (15)$$

$$c_T = \cos \Delta_T, \quad s_T = \sin \Delta_T, \quad \Delta_T = \varphi_T^p - \varphi_T^s$$

Please note, that this definition differs in two ways from the definition in Chipman (2009b): the retardance is defined differently there ($\Delta_X = \varphi_X^s - \varphi_X^p$), and we denote with D the

horizontal diattenuation parameter d_h (Chipman 2009b) and not the diattenuation magnitude $D_{mag} = |D|$ (see Suppl. S.4). The Müller matrix for the reflecting part of the PBS Eq. (16) includes a mirror reflection (Suppl. S.6) with the corresponding intensity reflection coefficients (reflectance) for light polarised parallel ($R_p = T_R^p$) and perpendicular ($R_s = T_R^s$) to the plane of incidence (Suppl. S.1) of the polarising beam-splitter.

$$\mathbf{M}_R = T_R \begin{pmatrix} 1 & D_R & 0 & 0 \\ D_R & 1 & 0 & 0 \\ 0 & 0 & -Z_R c_R & -Z_R s_R \\ 0 & 0 & Z_R s_R & -Z_R c_R \end{pmatrix} = T_R \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & D_R & 0 & 0 \\ D_R & 1 & 0 & 0 \\ 0 & 0 & Z_R c_R & Z_R s_R \\ 0 & 0 & -Z_R s_R & Z_R c_R \end{pmatrix} \quad (16)$$

$$T_R = \frac{T_R^p + T_R^s}{2}, \quad D_R = \frac{T_R^p - T_R^s}{T_R^p + T_R^s}, \quad Z_R = \frac{2\sqrt{T_R^p T_R^s}}{T_R^p + T_R^s} = \sqrt{1 - D_R^2}, \quad (17)$$

$$c_R = \cos \Delta_R, \quad s_R = \sin \Delta_R, \quad \Delta_R = \varphi_R^p - \varphi_R^s$$

In order to simplify the derivation of the equations, we describe both the reflecting and transmitting matrices with the matrix \mathbf{M}_S , and replace the subscript s (for splitter) by T (transmitting) or R (reflecting) where appropriate, which means

$$D_S \in \{D_R, D_T\}, \quad \mathbf{M}_S \in \{\mathbf{M}_R, \mathbf{M}_T\}, \quad I_S \in \{I_R, I_T\} \quad (18)$$

It has to be emphasised, that for this reason we can't use the diattenuation magnitude D_{mag} , which is always positive and almost exclusively used in other publications, but have to use the diattenuation parameter D , which changes the sign when T_R^s becomes larger than T_R^p (see S.3). Please keep also in mind that usually $D_R < 0$, that \mathbf{M}_R includes an additional mirror reflection, and that fluxes measured after the PBS are not influenced by the addition of an ideal mirror reflection in the optical path.

2.3 Calibration, linear depolarisation ratio, and total signal

Eq. (6) shows the Stokes vectors of the transmitted (I_T) and reflected (I_R) channels, alias I_S , after the polarising beam-splitter \mathbf{M}_S (PBS) without calibrator, i.e. $\mathbf{C} = \mathbf{1}$ = identity matrix. Eq. (6) represents the standard measurement at the axial rotation of 0° , neglecting for now additional optics in \mathbf{M}_O .

$$\begin{aligned} I_S(0^\circ) &= \eta_S \mathbf{M}_S \mathbf{F} I_L = \eta_S \mathbf{M}_S I_{in} = \\ &= \eta_S T_S \begin{pmatrix} 1 & D_S & 0 & 0 \\ D_S & 1 & 0 & 0 \\ 0 & 0 & Z_S c_S & Z_S s_S \\ 0 & 0 & -Z_S s_S & Z_S c_S \end{pmatrix} F_{11} I_L \begin{pmatrix} 1 \\ a \\ 0 \\ 0 \end{pmatrix} = \eta_S T_S F_{11} I_L \begin{pmatrix} 1 + D_S a \\ D_S + a \\ 0 \\ 0 \end{pmatrix} \end{aligned} \quad (19)$$

The measured signals I_S are

$$I_S(0^\circ) = \eta_S T_S F_{11} I_L (1 + D_S a) \quad (20)$$

which correspond to the transmitted and reflected intensities, include the individual channels gains η_S , i.e. η_T and η_R , which are the product of the electronic amplification of the detectors, the amplifiers, and of the optical attenuation due to polarisation insensitive attenuation of all optics including neutral density and interference filters. The latter is in general different in the

two channels. We can solve the equation of the ratio of the measured reflected to the transmitted signals

$$\frac{I_R(0^\circ)}{I_T} = \frac{\eta_R T_R (1 + D_R a)}{\eta_T T_T (1 + D_T a)} = \frac{\eta_R (T_R^p + T_R^s \delta)}{\eta_T (T_T^p + T_T^s \delta)} \quad (21)$$

for the linear depolarisation ratio δ if we know the calibration factor

$$\eta \equiv \frac{\eta_R T_R}{\eta_T T_T} \quad (22)$$

(with reflectance T_R and transmittance T_T for unpolarised light) and the transmission parameters of the polarising beam-splitter T_T^p , T_T^s , T_R^p , and T_R^s for the correction of its cross talk. We could get the calibration factor η already with the measurements in Eq. (21) if the light incident on the analyser was unpolarised, i.e. $a = 0$. Else, η can be determined by means of calibration measurements, e.g. by rotating the the PBS including the detectors by $+45^\circ$ or -45° about the optical axis (Eq. (23)).

$$\begin{aligned} I_S(\pm 45^\circ) &= \eta_S \mathbf{M}_S \mathbf{R}(\pm 45^\circ) \mathbf{F} I_m = \\ &= \eta_S \mathbf{M}_S \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \mp 1 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} F_{11} I_L \begin{pmatrix} 1 \\ a \\ 0 \\ 0 \end{pmatrix} = \eta_S T_S \begin{pmatrix} 1 & D_S & 0 & 0 \\ D_S & 1 & 0 & 0 \\ 0 & 0 & Z_S c_S & Z_S s_S \\ 0 & 0 & -Z_S s_S & Z_S c_S \end{pmatrix} F_{11} I_L \begin{pmatrix} 1 \\ 0 \\ \pm a \\ 0 \end{pmatrix} = \\ &= \eta_S T_S F_{11} I_L \begin{pmatrix} 1 \\ D_S \\ \pm a Z_S c_S \\ \mp a Z_S s_S \end{pmatrix} \end{aligned} \quad (23)$$

With the rotations $\mathbf{R}(\pm 45^\circ)$ it is intended to produce at the entrance of the PBS equal light intensities in the transmitted and reflected paths, independent of the atmospheric depolarisation. The error from an inaccurate $\pm 45^\circ$ alignment can be reduced by the $\Delta 90$ -calibration explained in Sect. 5. From Eq. (23) we get the signal intensities

$$I_S(\pm 45^\circ) = \eta_S T_S F_{11} I_L \quad (24)$$

and the calibration factor η from the signal ratio

$$\frac{I_R(\pm 45^\circ)}{I_T} = \frac{\eta_R T_R}{\eta_T T_T} = \eta \quad (25)$$

With known η we can express the measured signal ratio δ^* in Eq. (21) as

$$\delta^* \equiv \frac{1}{\eta} \frac{I_R(0^\circ)}{I_T} = \frac{I_T(\pm 45^\circ)}{I_R} \frac{I_R(0^\circ)}{I_T} = \frac{T_T T_R^p + T_R^s \delta}{T_R T_T^p + T_T^s \delta} \quad (26)$$

which is almost equal to the linear depolarisation ratio δ , but still includes the diattenuation and cross talk of the imperfect polarising beam-splitter. From δ^* we retrieve the linear depolarisation ratio δ

$$\delta = \frac{\delta^* T_R T_T^p - T_T T_R^p}{T_T T_R^s - \delta^* T_R T_T^s} \quad (27)$$

With the assumption for good PBSs

$$T_T^s \ll 1 \Rightarrow \left\{ T_R^s \approx 1, T_T \approx 0.5 T_T^p, T_R \approx 0.5(1 + T_R^p) \right\} \quad (28)$$

we get an approximation

$$\delta \approx \delta^* - T_R^p (1 - \delta^*) \quad (29)$$

Next we will determine the total lidar backscatter signal from the two signals I_T and I_R measured at 0° . This is the range dependent signal, which we use for the inversion of the backscatter coefficient F_{11} with the lidar inversion methods. From Eq. (20) we can get F_{11} either from the transmitted or from the reflected signal

$$F_{11} = \frac{I_S(0^\circ)}{\eta_S T_S I_L (1 + D_S a)} \quad (30)$$

The polarisation parameter a can be extracted from the signal ratio in Eq. (21)

$$a = \frac{\eta I_T - I_R}{I_R D_T - \eta I_T D_R}, \quad (31)$$

and substituted in Eq. (30) to yield

$$I_L F_{11} = \frac{\eta T_T D_T I_R - \eta_R T_R D_R I_T}{\eta_T T_T \eta_R T_R (D_T - D_R)} = \frac{1}{D_T - D_R} \left(\frac{D_T I_R}{\eta_R T_R} - \frac{D_R I_T}{\eta_T T_T} \right). \quad (32)$$

Equation (32) shows that we cannot determine an absolute F_{11} without an absolute calibration of the individual channel gains η_R and η_T and knowledge of the laser intensity I_L . However, for the lidar signal inversions, which use a reference value at a certain range or similar, we only need a relative, range dependent F_{11} . Hence we can choose any of the range independent parameters in Eq. (32), in which only I_T and I_R are range dependent, which we cancel and get

$$F_{11} \propto D_T I_R - \eta D_R I_T = \frac{T_T^p - T_T^s}{T_T^p + T_T^s} I_R - \eta \frac{T_R^p - T_R^s}{T_R^p + T_R^s} I_T. \quad (33)$$

In case the polarising beam-splitter is ideal, i.e. $T_T^p = T_R^s = 1$ and $T_T^s = T_R^p = 0$, and hence $D_R = -1$ and $D_T = +1$, Eq. (33) becomes as expected

$$F_{11} \propto I_R + \eta I_T, \quad (34)$$

Please bear in mind that in general $T_R^s > T_R^p$, and therefore $(T_R^p - T_R^s) < 0$ and $D_R < 0$ according to our definition in Eq. (17).

Summarising: we have to find the calibration factor η and correct the cross talk. δ is retrieved from two signals at 0° represented by δ^* , Eq. (26), plus two signals for the calibration factor at $\pm 45^\circ$, Eq. (25), and the knowledge of the PBS parameters T_T^p , T_T^s , T_R^p , and T_R^s for the correction of the cross talk.

3 Complete Müller-Stokes lidar setup with rotation of optical elements

In the previous section, a basic lidar setup is described with the Müller-Stokes formalism as an introduction, which includes only a horizontal-linear polarised laser, the matrices for the atmospheric aerosol backscattering and depolarisation, and the polarising beam-splitter. In order to expand this setup to a realistic but still manageable model for a large variety of lidar systems and calibration techniques, we introduce in this section some concepts and variables, which will enable us to describe the variety of setups with as few as possible equations.

The Stokes-Müller formalism (Chipman, 2009b) represents four linear equations (Eq. (35)), which relate the four output with the four input Stokes parameters.

$$\begin{aligned} \mathbf{I}_{out} &= \begin{pmatrix} I_{out} \\ Q_{out} \\ U_{out} \\ V_{out} \end{pmatrix} = \mathbf{M} \mathbf{I}_{in} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \begin{pmatrix} I_{in} \\ Q_{in} \\ U_{in} \\ V_{in} \end{pmatrix} = \\ &= M_{11} I_{in} \begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{pmatrix} \end{aligned} \quad (35)$$

The small letter matrix (m_{ij}) and vector components at the right of Eq. (35) are normalised by their first element, i.e. the unpolarised transmission M_{11} and the total intensity I_{in} ; hence $m_{11} = i_{in} = 1$. In the following we usually keep the variable i_{in} in order to allow for later expansions of the equations. While the first Stokes vector parameter I_{out} can be directly detected with a photon detector, the other output Stokes parameters can each be determined with two measurements of output intensities using additional polarisation elements (Chipman 2009a) (see Eq. S.2.2). We derive the backscatter coefficient F_{11} and the linear polarisation parameter a of the Müller matrix \mathbf{F} of the atmosphere (see Sect. 2.1) from the first two equations of I_{out} and Q_{out} in Eq. (35), which in turn are determined from the two measurements of I_R and I_T using the two orthogonal linear analysers of the polarising beam-splitter. For the determination of each additional unknown parameter we need additional measurements. For the relative calibration factor η of the two polarisation signals I_R and I_T we use an additional calibrator element with Müller matrix \mathbf{C} . The lidar setup shown in Fig (1) is described by Eq. (6), i.e. $\mathbf{I}_S = \eta_S \mathbf{M}_S \mathbf{C} \mathbf{M}_O \mathbf{F} \mathbf{M}_E \mathbf{I}_L$, where the matrices \mathbf{M}_{TR} (alias \mathbf{M}_S) represent the two paths of the polarising beam-splitter, i.e. subscripts T for transmission and R for reflection. Since the laser in our model can be arbitrarily polarised and because "parallel" and "perpendicular" are defined relative to the incident plane of a beam-splitter (superscripts p and s , respectively; see Suppl. S.1) and don't necessarily describe the polarisation behind it with respect to the laser polarisation, we can't use these terms here for the two branches behind the polarising beam-splitter. $\mathbf{C}(\Psi)$ describes the calibrator matrix, which can be a mechanical rotation of the detection optics by Ψ or an optical device as a polarising sheet filter rotated by angle Ψ , for example. The purpose of the calibrator device is to produce equal intensities for both polarisation channels, independent of the laser light polarisation and independent of backscattering characteristics of the atmosphere. This is e.g. achieved with an ideal polarising sheet filter oriented at 45° with respect to the incident plane of the PBS. The calibration factor η of the relative sensitivity of both polarisation channels can be retrieved from the ratio of the measured intensities. The calibration factor includes electronic gains and the polarisation

transmission of optical elements behind the calibrator. In our model the calibrator can be at three different positions in the optical chain, which are indicated by the red blocks in Fig. (2). The calibrator positions and the respective equations are these:

$$\text{behind the laser emitter optics } \mathbf{M}_E \quad I_S = \eta_S \mathbf{M}_S \mathbf{M}_O \mathbf{F} \mathbf{C} \mathbf{M}_E I_L = \eta_S \mathbf{M}_S \mathbf{M}_O \mathbf{F} \mathbf{C} I_{in} \quad (36)$$

$$\text{before the telescope / receiver optics } \mathbf{M}_O \quad I_S = \eta_S \mathbf{M}_S \mathbf{M}_O \mathbf{C} \mathbf{F} \mathbf{M}_E I_L = \eta_S \mathbf{M}_S \mathbf{M}_O \mathbf{C} I_{in} \quad (37)$$

$$\text{before the polarising beam-splitter } \mathbf{M}_S \quad I_S = \eta_S \mathbf{M}_S \mathbf{C} \mathbf{M}_O \mathbf{F} \mathbf{M}_E I_L = \eta_S \mathbf{M}_S \mathbf{C} I_{in} \quad (38)$$

In case the telescope and/or the collimating lens don't change the state of polarisation of the incoming light, the placement of the calibrator after those elements is equivalent to the position before the telescope.

We develop the equations for all three positions of the calibrator, and additionally for the calibration with an unpolarised light source before the receiving optics (Sect. 6). In the equations we use as calibrator elements the Müller matrix \mathbf{C} as a place holder for any sort of calibrator, which are \mathbf{M}_{rot} for mechanical rotation or by means of a $\lambda/2$ plate, \mathbf{M}_P for a linear polariser, \mathbf{M}_{QW} for a $\lambda/4$ plate, and \mathbf{M}_{CP} for a circular polariser.

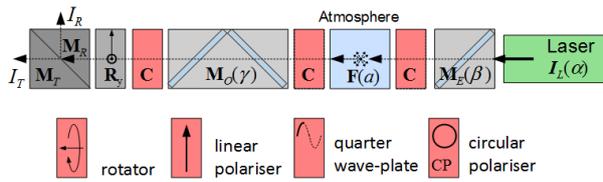


Figure 2: Schematic of a 2-channel, polarisation sensitive lidar setup (compare Fig. 1) with Müller matrix block elements and different calibrator (red block) positions (top), and three options for the calibrator \mathbf{C} (bottom). I_L : laser Stokes vector, \mathbf{M}_E : emitter optics; \mathbf{F} : atmospheric backscatter matrix with polarisation parameter a ; \mathbf{M}_O : receiver optics; \mathbf{R}_y : rotation matrix for the 0° ($y=+1$) and 90° ($y=-1$) detection setup (see text); $\mathbf{M}_{T,R}$: transmitted and reflected part of the polarising beam-splitter; $I_{T,R}$: transmitted and reflected detection signals. Angles α , β , and γ are rotations around the optical axis.

3.1 The analyser $\langle \text{bra} |$ and input $| \text{ket} \rangle$ vectors

The general structure of all the considered lidar setups can be described with three groups of optical elements: elements before the calibrator, the calibrator, and elements behind the calibrator. To simplify the equations, we combine the matrices after the calibrator to an analyser matrix \mathbf{A}_S , and the matrices before the calibrator together with the Stokes vector of the laser beam I_L to an input Stokes vector I_{in} . Since \mathbf{A}_S and I_{in} are the same for all calibrator types, they have to be derived only once and can then be used for the different setups. "After" and "before" denote the order with respect to the light direction, i.e. from right to left in the Müller-Stokes equations.

Since photo detectors are, in general, insensitive to the polarisation, we measure the intensity I_S at the detector, which is the first parameter of the output Stokes vector. I_S is determined by the top row of a matrix \mathbf{A}_S and an input vector I_{in} .

$$\begin{pmatrix} I_S \\ - \\ - \\ - \end{pmatrix} = \eta_S \mathbf{A}_S I_{in} = \eta_S \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix} \begin{pmatrix} I_{in} \\ Q_{in} \\ U_{in} \\ V_{in} \end{pmatrix} = \eta_S \begin{pmatrix} A_{11}I_{in} + A_{12}Q_{in} + A_{13}U_{in} + A_{14}V_{in} \\ - \\ - \\ - \end{pmatrix} \quad (39)$$

Using the $\langle \text{bra} | \text{ket} \rangle$ matrix-vector notation (see App. B and App. D), we define for this work the row vector $\langle \mathbf{A}_S |$ as the top row of a matrix \mathbf{A}_S ,

$$\langle \mathbf{A}_S | = \langle A_{11} \quad A_{12} \quad A_{13} \quad A_{14} | \quad (40)$$

and use analogously the column vector $| \mathbf{I}_{in} \rangle$. With this notation the equation for the intensity I_S can be written as

$$\begin{aligned} I_S &= \eta_S \langle \mathbf{A}_S | \mathbf{I}_{in} \rangle = \eta_S I_{in} \langle A_{11} \quad A_{12} \quad A_{13} \quad A_{14} | I_{in} \quad Q_{in} \quad U_{in} \quad V_{in} \rangle = \\ &= \eta_S I_{in} (A_{11} I_{in} + A_{12} Q_{in} + A_{13} U_{in} + A_{14} V_{in}) \end{aligned} \quad (41)$$

For example, the equation for signal I_S of a calibration measurement with the calibrator before the PBS (see Eq. (38)) can be expressed as

$$I_S(y, x, \varepsilon) = \eta_S \langle \mathbf{M}_S \mathbf{R}_y | \mathbf{C}(x45^\circ + \varepsilon) | \mathbf{M}_O \mathbf{F} \mathbf{M}_E \mathbf{I}_L \rangle = \eta_S \langle \mathbf{A}_S | \mathbf{C} | \mathbf{I}_{in} \rangle \quad (42)$$

and the respective standard measurement signals without the calibrator can be expressed with the same vectors $\langle \mathbf{A}_S |$ and $| \mathbf{I}_{in} \rangle$ as

$$I_S(y) = \eta_S \langle \mathbf{M}_S \mathbf{R}_y | \mathbf{M}_O \mathbf{F} \mathbf{M}_E \mathbf{I}_L \rangle = \eta_S \langle \mathbf{A}_S | \mathbf{I}_{in} \rangle \quad (43)$$

This split-up of the equations in an analyser bra-vector and an input Stokes ket-vector is similar to the split-up in instrumental vectors of the transmitter and receiver in Kaul et al. (2004) and Volkov et al. (2015).

In Eqs. (42) and (43) we already used the binary operators y , x , the parameter ε for different rotation angles, and the rotation matrix \mathbf{R}_y , which will be explained in detail in Sect. 3.3.

3.2 Laser polarisation and atmospheric depolarisation

The light leaving commercial Nd:YAG lasers is usually linearly polarised. Manufacturers often specify a polarisation "purity" $> 95\%$ or similar, which is not very accurate. Actually, the laser light is often much better polarised, but the measurement of the polarisation of individual lasers in a series is expensive and it can change during the operation and with ageing of the laser. Probably for that reason the manufacturers seem to specify a lower limit which they can assure under all circumstances. A secure method to ensure a high degree of linear polarisation is to use a polariser as the last element at the laser output. Often the orientation of the laser polarisation relative to the orientation of the polarising beam-splitter in the receiving optics is not well known, first, because the state of polarisation of short laser pulses with high power is difficult to measure accurately, and second, the state of polarisation of the laser can change during the operation of the laser over periods with changing environmental conditions. Hence we consider a possible rotation α of the plane of horizontal-linear polarisation of the laser (laser rotation). Furthermore, beam expanders and especially steering mirrors after the laser can degrade the degree of linear polarisation considerably producing elliptical polarised light. Hence we start with an emitter Stokes vector with arbitrary state of polarisation leaving the laser, which includes all effects of cleaning, shaping and steering optics

$$\mathbf{I}_E = \mathbf{M}_E \mathbf{I}_L = T_E I_L | i_E \quad q_E \quad u_E \quad v_E \rangle \quad (44)$$

We will develop all equations first for a general emitter beam polarisation as in Eq. (44), and then as an explicit example for a linearly polarised laser with intensity I_L and laser rotation α

(see App. D) to elaborate the errors due to misalignments of the calibration and measurement optics.

$$\mathbf{I}_L(\alpha) = I_L |1 \quad c_{2\alpha} \quad s_{2\alpha} \quad 0\rangle \quad (45)$$

Depolarisation of the laser (with linear polarisation parameter a_L), caused by volume or surface scattering in or on optical elements, is hardly probable, and the scattered radiation reaching the lidar telescope would be negligible. However, it is briefly treated in Suppl. S.3. The Stokes vector \mathbf{I}_F , which is reflected by the atmosphere with scattering matrix $\mathbf{F}(a)$ with linear polarisation parameter a from a generally polarised emitter \mathbf{I}_E , is (see Suppl. S.3)

$$\frac{\mathbf{I}_F(a)}{F_{11}T_E I_L} = \frac{\mathbf{F}(a)|\mathbf{M}_E \mathbf{I}_L\rangle}{F_{11}T_E I_L} = |i_E \quad aq_E \quad -au_E \quad (1-2a)v_E\rangle \quad (46)$$

3.3 Receiver optics and calibrator

In order to investigate the effect of misalignments of the optical elements on the final measurement and the calibration results, i.e. the total signal and the linear depolarisation ratio, we apply to each optical element in Eqs. (36) to (38) an additional rotation error about the optical axis (see Fig. (2)). The reference coordinate system is in general defined by the incident plane of the polarising beam-splitter (Fig. (3)), wherefore no rotation error is considered in \mathbf{M}_S . Nevertheless, the polarising beam-splitter can be mechanically rotated by 90° in some existing lidar systems without changing the rest of the setup. We include this additional fixed rotation by introducing the rotation matrix \mathbf{R}_y with the polarising beam-splitter orientation parameter y (Fig. (3)). For $y = +1$ the parallel laser polarisation is detected in the transmitted channel and for $y = -1$ in the reflected channel. This seems a bit confusing, but it is necessary to get control of all the actually existing lidar setups. The rotation matrix \mathbf{R}_y is shown in Eq. (47).

$$\mathbf{R}_y = \mathbf{R}(y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} \mathbf{R}(y = -1) = \mathbf{R}(90^\circ) \\ \mathbf{R}(y = +1) = \mathbf{R}(0^\circ) \end{cases} \quad (47)$$

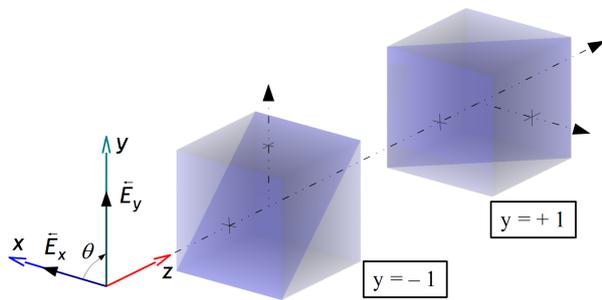
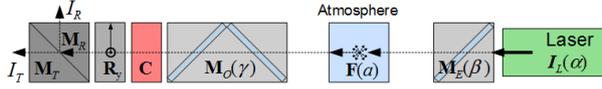


Figure 3: Definition of the global coordinate reference system and the binary operator y with respect to the incident plane of the polarising beam-splitter. If the polarising beam-splitter orientation parameter $y = +1$, the vibration of the horizontal-linear polarisation with vector \mathbf{E}_x is parallel to the plane of incidence, while for $y = -1$ it is perpendicular.

The whole lidar system shown in Fig.(2) is then described by Eq. (48) with rotation angles α , β , γ , and Ψ around the optical axis.

$$\mathbf{I}_S(y, \Psi, \gamma, a, \beta, \alpha) = \eta_S \mathbf{M}_S \mathbf{R}(y) \mathbf{C}(\Psi) \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha) \quad (48)$$



It would be possible to include the \mathbf{R}_y rotation by changing the laser angle α in Eq. (48), but we choose to do it before the polarising beam-splitter for two reasons: first we want to use the angle α only for rotation errors, and second, in some lidar systems a rotation of the receiving optics is used for the calibration, and with these setups a change between the two \mathbf{R}_y versions of a lidar is easily accomplished and can be used for certain test measurements without changing the rest of the equations. On the other hand, an arbitrary rotation of the laser polarisation is usually not possible. A rotation γ of a retarding diattenuator \mathbf{M}_O can complicate the equations considerably, as it converts linearly polarised light into elliptically polarised, which cannot be analysed by a simple polarising beam-splitter. Therefore, diattenuating and retarding optics before the polarising beam-splitter should be carefully oriented with their eigen axes parallel to the ones of the polarising beam-splitter to avoid the resulting uncertainties. Such an element can e.g. be a dichroic beam-splitter, which does not reflect exactly to 0° or 90° . For what we call $\Delta 90$ -calibration, we use two calibrator orientations $\mathbf{C}(\Psi)$ with

$$\Psi^+ = +45^\circ + \varepsilon \quad (49)$$

$$\Psi^- = -45^\circ + \varepsilon$$

so that

$$\Psi^+ - \Psi^- = 90^\circ \quad (50)$$

We choose these special angles, because in the geometric mean of two calibrations at orientations exactly 90° apart the error terms sometimes compensate very well. Note, that the $\Delta 90$ error angle ε describes the rotational misalignment of the whole $\Delta 90$ -calibrator setup with respect to the polarising beam-splitter, not the error in the 90° difference. So, $\pm 45^\circ$ means either $+45^\circ$ or -45° , and $\Delta 90$ means the combination of measurements at $+45^\circ + \varepsilon$ and $-45^\circ + \varepsilon$. To obtain general equations, we combine these angles using the binary operator x for calibrations

$$x = \pm 1: \Psi(x, \varepsilon) = x45^\circ + \varepsilon \quad (51)$$

We use this definition in a setup with a rotation calibrator \mathbf{M}_{rot} (Sect. 7)

$$\mathbf{C}(\Psi, h) = \mathbf{M}_{rot}(x45^\circ + \varepsilon, h) = \mathbf{M}_{rot}(x, \varepsilon, h) = \mathbf{R}(x45^\circ) \mathbf{R}(\varepsilon) \mathbf{M}_h \quad (52)$$

with the binary operator h to discern between a mechanical ($h = +1$) and a $\lambda/2$ plate rotation (Suppl. S.10.15), and can express the four equations for the reflected and transmitted signals I_R and I_T of the two calibration measurements at $\Psi = \pm 45^\circ + \varepsilon$ with the one formula Eq. (53)

$$I_S(y, x, \varepsilon, h, \gamma, a, \beta, \alpha) = \eta_S \langle \mathbf{M}_S \mathbf{R}_y | \mathbf{M}_{rot}(x, \varepsilon, h) | \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha) \rangle \quad (53)$$

and the four equations for the standard measurements at $\Psi = 0^\circ$ ($y = +1$) and $\Psi = 90^\circ$ ($y = -1$) using the same analyser and input Stokes vectors with just another formula Eq. (54)

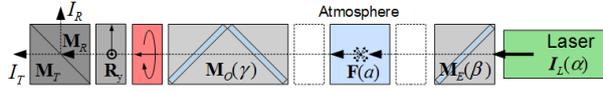
$$I_S(y, \varepsilon, h, \gamma, a, \beta, \alpha) = \eta_S \langle \mathbf{M}_S \mathbf{R}_y | \mathbf{R}(\varepsilon) \mathbf{M}_h | \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha) \rangle \quad (54)$$

Using the rotation calibrator we have to consider the same alignment error ε for the standard measurements at 0° and 90° as for the calibration at the $\pm 45^\circ$, because this calibrator is not

removed from the lidar setup after the calibration measurements. Please note that $\varepsilon = 0$ for all other calibrators.

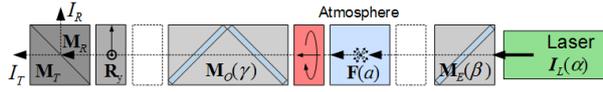
4 Retrieval of the total signal and of the linear depolarisation ratio

The final goal of this work is to investigate how the polarisation calibration factor, the linear depolarisation ratio, and the total lidar signal can be retrieved from the measurements I_T and I_R , how much the various rotational misalignments and the crosstalk of the calibrator influence them, and how the deviations can possibly be corrected. The standard measurement signals I_S in Eq. (54) include a rotational error ε before the polarising beam-splitter.



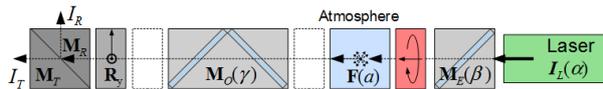
We get Eq. (55) for the analyser part with Eqs. (D.5), (Suppl. S.5.1.6 and S.10.15.2), and with the most general input I_E from Eq. (E.31) with atmospheric polarisation parameter a we get the signal I_S from Eq. (S.7.1.2)

$$\begin{aligned} \langle \mathbf{A}_S(y) | \mathbf{R}(\varepsilon, h) \rangle &= \langle \mathbf{M}_S \mathbf{R}_y | \mathbf{R}(\varepsilon) \mathbf{M}_h \rangle = T_S \langle 1 \quad y c_{2\varepsilon} D_S \quad -y h s_{2\varepsilon} D_S \quad 0 \rangle \\ \frac{I_S}{\eta_S T_S T_{rot} T_O F_{11} T_E I_L} &= \frac{\langle \mathbf{A}_S(y) | \mathbf{R}(\varepsilon) \mathbf{M}_h | I_{in}(\gamma, a) \rangle}{T_S T_{rot} T_O F_{11} T_E I_L} = \frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{R}(\varepsilon) \mathbf{M}_h | \mathbf{M}_O(\gamma) \mathbf{F}(a) I_E \rangle}{T_S T_{rot} T_O F_{11} T_E I_L} = \\ &= \left(1 + y D_S D_O c_{2\gamma+h2\varepsilon} \right) i_E - y D_S Z_O S_O s_{2\gamma+h2\varepsilon} v_E + \\ &+ a \left\{ D_O \left(c_{2\gamma} q_E - s_{2\gamma} u_E \right) + y D_S \left[\left(c_{2\varepsilon} q_E + s_{h2\varepsilon} u_E \right) - s_{2\gamma+h2\varepsilon} \left(W_O \left(s_{2\gamma} q_E + c_{2\gamma} u_E \right) - 2 Z_O S_O v_E \right) \right] \right\} \end{aligned} \quad (56)$$



In case the rotational error is before the receiving optics, we get Eq. (57) from Eq. (S.7.2.1) with Eq. (D.7) for the analyser part and (E.26) for the input vector.

$$\begin{aligned} \frac{I_S}{\eta_S T_S T_O T_{rot} F_{11} T_E I_L} &= \frac{\langle \mathbf{A}_S(y, \gamma) | I_{in, \varepsilon}(\varepsilon, h, a) \rangle}{T_S T_O T_{rot} F_{11} T_E I_L} = \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) | \mathbf{R}(\varepsilon) \mathbf{M}_h \mathbf{F}(a) I_E \rangle}{T_S T_O T_{rot} F_{11} T_E I_L} = \\ &= \left(1 + y D_O D_S c_{2\gamma} \right) i_E - y D_S Z_O S_O s_{2\gamma} h v_E + \\ &+ a \left\{ D_O \left[c_{2\gamma-2\varepsilon} q_E - s_{2\gamma-2\varepsilon} h u_E \right] + \right. \\ &\left. + y D_S \left[\left(c_{2\varepsilon} q_E + s_{2\varepsilon} h u_E \right) - s_{2\gamma} \left(W_O \left(s_{2\gamma-2\varepsilon} q_E + c_{2\gamma-2\varepsilon} h u_E \right) - 2 Z_O S_O h v_E \right) \right] \right\} \end{aligned} \quad (57)$$



The case of rotational error behind the emitter optics can be retrieved from Eq. (57) by simply replacing ε with $-\varepsilon$ (see Suppl. S.7.3). Special cases of I_E for Eqs. (56) and (57) can be found in Sect. E.2.

4.1 General formulations for the total signal and the linear depolarisation ratio

From Eqs. (56) and (57) we see that all standard signals I_S can be expressed by introducing two parameters G_S and H_S for the terms without and with atmospheric polarisation, respectively,

$$I_S = \eta_S T_S T_O T_{rot} F_{11} T_E I_L (G_S + a H_S) \quad (58)$$

Using Eq. (56) as an example, the two parameters are

$$\begin{aligned} G_S(y, \varepsilon, h, \gamma) &= (1 + y D_S D_O c_{2\gamma+h2\varepsilon}) i_E - y D_S Z_O s_O s_{2\gamma+h2\varepsilon} v_E \\ H_S(y, \varepsilon, h, \gamma, \beta, \alpha) &= \\ &= D_O (c_{2\gamma} q_E - s_{2\gamma} u_E) + y D_S \left[(c_{2\varepsilon} q_E + s_{h2\varepsilon} u_E) - s_{2\gamma+h2\varepsilon} \left(W_O (s_{2\gamma} q_E + c_{2\gamma} u_E) - 2 Z_O s_O v_E \right) \right] \end{aligned} \quad (59)$$

Table 1 shows how their expressions simplify if some uncertainties are neglected.

With Eq. (58) the measured signal ratio becomes

$$\delta^* = \frac{1}{\eta} \frac{I_R}{I_T} = \frac{G_R + a H_R}{G_T + a H_T} \quad (60)$$

with the calibration factor $\eta = (\eta_R T_R) / (\eta_T T_T)$, which has to be determined with one of the methods in the following chapters. G_S and H_S describe the polarisation cross-talk terms of the lidar setup depending on the diattenuation parameters D and the retardation (described by s_O and c_O) of the individual optical elements, depending on the relative rotation of the elements and on the polarisation parameter of the atmosphere a . From Eq. (60) we retrieve the general equations for the polarisation parameter a in Eq. (61) and for the linear depolarisation ratio δ in Eq. (62) (compare Eq. (12)).

$$a = \frac{\delta^* G_T - G_R}{H_R - \delta^* H_T} \quad (61)$$

$$\delta = \frac{1-a}{1+a} = \frac{\delta^* (G_T + H_T) - (G_R + H_R)}{(G_R - H_R) - \delta^* (G_T - H_T)} \quad (62)$$

Remind that δ^* and hence a and δ are range dependent. For the retrieval of the total lidar signal, which is equivalent to F_{11} , we substitute Eq. (61) in Eq. (58) in the transmitted or the reflected version of $I_S \in \{I_T, I_R\}$ and replace δ^* by Eq. (60). Using the transmitted signal I_T from Eq. (58) we get Eq. (63), and after some restructuring (see Eqs.(S.8.1) and (S.8.2)) we get the attenuated backscatter coefficient Eq. (64).

$$\eta_T T_T T_O F_{11} T_E I_L = \frac{I_T}{G_T + a H_T} \quad (63)$$

$$F_{11} = \frac{1}{T_O T_E I_L} \frac{H_R \frac{I_T}{\eta_T T_T} - H_T \frac{I_R}{\eta_R T_R}}{H_R G_T - H_T G_R} \quad (64)$$

For the inversion of the lidar signal we only need the relative attenuated backscatter coefficient, for which we can get a much simpler formula by removing all factors in Eq. (64) which are not range dependent (compare Eq. 32 ff), which yields Eq.(65)

$$F_{11} \propto \eta H_R I_T - H_T I_R \quad (65)$$

The individual calibration methods can add errors and uncertainties due to additional optics with unknown diattenuation and retardation and due to rotation errors. The possible uncertainties of the calibration factor η can be assessed from the analytical expressions of the gain ratio η^* (see Sect.(5)).

For systems without a polarising beam-splitter, i.e. pure backscatter lidars with one channel for each wavelength, the total signal is I_T from the transmitted signal, but with $D_S = D_T = 0$, and without calibrator ($\Rightarrow h = 1$) and without calibrator rotation error angle ε . Hence, we get from both Eqs. (56) and (57) the transmitted signal with Eq. (66)

$$D_T = 0, T_T = 1, \varepsilon = 0, y = 1 \Rightarrow$$

$$I_T = \eta_T T_T T_O F_{11} T_E I_L \left[i_E + a D_O (c_{2\gamma} q_E - s_{2\gamma} u_E) \right] \quad (66)$$

which shows that there is a distortion of the total signal due to the receiver optics diattenuation and depending on the atmospheric depolarisation, even if the laser beam behind the emitter optics is perfectly horizontal-linearly polarised and without receiver optics rotation. i.e. Eq. (66) with

$$\gamma = 0, T_E = 1, i_E = q_E = 1, u_E = 0 \Rightarrow$$

$$I_T = \eta_T T_O F_{11} I_L [1 + a D_O] \quad (67)$$

Table 1. Parameters G_S and H_S and their simplifications when neglecting some uncertainties for the case of Eq.(56), i.e. rotational error ε before the polarising beam-splitter, - with a general emitter Stokes vector $\mathbf{I}_E = T_E I_L |i_E \ q_E \ u_E \ v_E\rangle$ (see App. E.2).

	G_S	H_S	
General	$(1 + yD_S D_O c_{2\gamma + h2\varepsilon}) i_E - yD_S Z_O s_{2\gamma + h2\varepsilon} v_E$	$D_O (c_{2\gamma} q_E - s_{2\gamma} u_E) + yD_S [(c_{2\varepsilon} q_E + s_{h2\varepsilon} u_E) - s_{2\gamma + h2\varepsilon} \{W_O (s_{2\gamma} q_E + c_{2\gamma} u_E) - 2Z_O s_O v_E\}]$	(68)
$\varepsilon = 0$	$(1 + yD_S D_O c_{2\gamma}) i_E - yD_S Z_O s_{2\gamma} v_E$	$D_O (c_{2\gamma} q_E - s_{2\gamma} u_E) + yD_S [q_E - s_{2\gamma} \{W_O (s_{2\gamma} q_E + c_{2\gamma} u_E) - 2Z_O s_O v_E\}]$	(69)
$\gamma = 0$	$(1 + yD_S D_O c_{h2\varepsilon}) i_E - yD_S Z_O s_{h2\varepsilon} v_E$	$D_O q_E + yD_S [c_{2\varepsilon} q_E + s_{h2\varepsilon} Z_O (c_O u_E + 2s_O v_E)]$	(70)
$\gamma = \varepsilon = 0$	$(1 + yD_S D_O) i_E$	$(D_O + yD_S) q_E$	(71)
$D_O = W_O = s_O = 0$	1	$yD_S (c_{2\varepsilon} q_E + s_{h2\varepsilon} u_E)$	(72)
- with a rotated, linearly polarised emitter Stokes vector $\mathbf{I}_E = T_E I_L i_E \ q_E \ u_E \ v_E\rangle = T_E I_L 1 \ c_{2\alpha} \ s_{2\alpha} \ 0\rangle$			
	G_S	H_S	
General	$1 + yD_S D_O c_{2\gamma + h2\varepsilon}$	$D_O c_{2\alpha + 2\gamma} + yD_S (c_{2\alpha - 2\varepsilon} - s_{2\gamma + h2\varepsilon} s_{2\alpha + 2\gamma} W_O)$	(73)
$\alpha = \varepsilon = 0$	$1 + yD_S D_O c_{2\gamma}$	$D_O c_{2\gamma} + yD_S (1 - s_{2\gamma}^2 W_O)$	(74)
$\gamma = 0$	$1 + yD_S D_O c_{h2\varepsilon}$	$D_O c_{2\alpha} + yD_S (c_{2\alpha - 2\varepsilon} - s_{2\alpha} s_{h2\varepsilon} W_O)$	(75)
$\alpha = \gamma = 0$	$1 + yD_S D_O c_{h2\varepsilon}$	$D_O + yD_S c_{2\varepsilon}$	(76)
$\gamma = \varepsilon = 0$	$1 + yD_S D_O$	$(D_O + yD_S) c_{2\alpha}$	(77)
$\alpha = \gamma = \varepsilon = 0$	$1 + yD_S D_O$	$D_O + yD_S$	(78)
$D_O = W_O = 0$	1	$yD_S c_{2\alpha - 2\varepsilon}$	(79)

5 The 45° and Δ90 calibration, the gain ratios and calibration factor

The measured, apparent calibration factor η^* of the polarisation channels, which we call in the following gain ratio in contrast to the calibration factor η , can be determined from the two calibration signals I_S , i.e. I_T and I_R , with a calibrator at $+45^\circ$ or -45° , which we call 45°-calibration (Eq. (80)). The calibration factor η is not directly measurable. Hence we need equations to retrieve η from the measured η^* .

$$\left. \begin{aligned} \eta^*(+45^\circ) &= \frac{I_R(+45^\circ)}{I_T(+45^\circ)} \\ \eta^*(-45^\circ) &= \frac{I_R(-45^\circ)}{I_T(-45^\circ)} \end{aligned} \right\} \rightarrow \eta^* = \frac{I_R}{I_T}(x45^\circ) \quad (80)$$

η^* includes alignment errors and cross talks. The theoretical dependence of these errors and cross-talks on the known parameters of our lidar model (Fig. 1) can be determined using the analytical expressions of Eqs. (81) and (82).

$$I_S(y, x45^\circ + \varepsilon) = \eta_S \langle \mathbf{A}_S(y) | \mathbf{C}(x45^\circ + \varepsilon) | \mathbf{I}_{in} \rangle \quad (81)$$

$$\eta^* = \frac{I_R(y, x45^\circ + \varepsilon)}{I_T(y, x45^\circ + \varepsilon)} = \frac{\eta_R \langle \mathbf{A}_R(y) | \mathbf{C}(x45^\circ + \varepsilon) | \mathbf{I}_{in} \rangle}{\eta_T \langle \mathbf{A}_T(y) | \mathbf{C}(x45^\circ + \varepsilon) | \mathbf{I}_{in} \rangle} \quad (82)$$

The theoretical correction K of the gain ratio to get the calibrator factor can be retrieved from the analytical expression Eq. (83), which is then used to correct the measurement in Eqs. (84) or (85).

$$K = \frac{\eta^*}{\eta} = \eta^* \frac{\eta_T T_T}{\eta_R T_R} = \frac{T_T \langle \mathbf{A}_R(y) | \mathbf{C}(x45^\circ + \varepsilon) | \mathbf{I}_{in} \rangle}{T_R \langle \mathbf{A}_T(y) | \mathbf{C}(x45^\circ + \varepsilon) | \mathbf{I}_{in} \rangle} \quad (83)$$

$$\eta = \frac{1}{K} \eta^* = \frac{1}{K} \frac{I_R}{I_T}(x45^\circ) \quad (84)$$

Furthermore, additional equations for the estimation of the uncertainty of η can be derived from Eq. (83). Since the errors due to ε cancel very well at orientations of the calibrator exactly $\Delta 90$ apart (i.e. $x = \pm 1$), as we will see in the following sections, a better estimation of the gain ratio can be retrieved from the geometric mean of the two gain ratios at $\pm 45^\circ$ (Eq. (85)), which we call $\Delta 90$ -calibration. The method of 90° different polariser angles to reduce errors in polarimetric measurements seems to be common in ellipsometry (Nee 2006).

$$\eta_{\Delta 90}^* \equiv \sqrt{\eta^*(+45^\circ + \varepsilon) \eta^*(-45^\circ + \varepsilon)} = \sqrt{\frac{I_R(+45^\circ + \varepsilon)}{I_T(+45^\circ + \varepsilon)} \cdot \frac{I_R(-45^\circ + \varepsilon)}{I_T(-45^\circ + \varepsilon)}} \quad (85)$$

While the two calibration signals I_T and I_R are taken at the same time, the two measurements for the $\Delta 90$ -calibration at $x45^\circ + \varepsilon$ are done subsequently, and the atmosphere can change in between. If the gain ratio η^* in Eq. (82) depends on the atmospheric polarisation parameter a , the $\Delta 90$ gain ratio $\eta_{\Delta 90}^*$ in Eq. (85) depends also on the temporal change of a . In order to avoid this dependency, we either have to choose an appropriate setup and adjust it so that η^* doesn't depend on a , or we have to choose a calibration range in which a doesn't change with time. In the following we assume the latter, i.e. that the atmospheric polarisation parameter a

does not change in the calibration range between the two calibration measurements at $x45^\circ+\varepsilon$. This does not mean that the backscatter coefficient, an extrinsic parameter, must not change, but only that the aerosol composition with its intrinsic parameter a stays the same and that the contribution of the air molecules to a is negligible. Nevertheless, in Sect. 11 we describe a method to determine and consequently correct for ε , which is one of the major factors in the a -dependency of η^* . In the following sections 6 to 10 we derive \mathbf{A}_S and \mathbf{I}_{in} for several positions of the calibrator \mathbf{C} , and with that we will analyse special cases of the measurements I_S and the retrieved calibration factor η . Figure (4) shows the steps in which the measurements are corrected for systematic errors by means of the model. If all system parameters of the model (Eqs. (56) and (57)) are known, the cross-talk parameters G_S and H_S can be calculated (see Eqs. (68) to (79)) and we only need to determine the calibration factor η by means of calibration measurements in step 2 and its correction for cross talk errors (step 3) as explained in Sects. 6 to 10. Under certain conditions some instrumental parameters can be determined by means of additional calibration measurements (step 4) described under “special cases” in Sects. 6 to 10 and in Sects. 11 and 12.

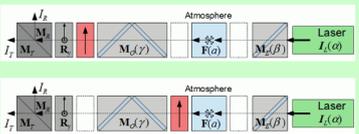
measurements	model
1. standard measurements	corrections (Sect. 4)
$I_S = \eta_S \langle \mathbf{A}_S \mathbf{I}_{in} \rangle \Rightarrow$	$I_S = \eta_S T_S T_O T_{rot} F_{11} T_E I_L (G_S + a H_S)$ $F_{11} \propto \eta H_R I_T - H_T I_R; \quad \eta = \frac{\eta_R T_R}{\eta_T T_T}$ $\delta = \frac{\delta^* (G_T + H_T) - (G_R + H_R)}{(G_R - H_R) - \delta^* (G_T - H_T)}; \quad \delta^* = \frac{1}{\eta} \frac{I_R}{I_T}$
2. calibration => gain ratio (Sect. 5)	
$I_S(x, \varepsilon) = \eta_S \langle \mathbf{A}_S \mathbf{C}(x, \varepsilon) \mathbf{I}_{in} \rangle \Rightarrow \eta_{A90}^*$	
3.	gain ratio correction => calibration factor (Sects. 6 to 10)
$\eta_{A90}^* \Rightarrow$	$\eta = \frac{\eta_{A90}^*}{K}$
4. combined calibration measurements	determination of instrumental parameters
$I_S(x, \varepsilon) = \eta_S \langle \mathbf{A}_S \mathbf{C}(x, \varepsilon) \mathbf{I}_{in}(\alpha) \rangle$ 	D_O (Sects. 7.2, 8.2) v_{in}, v_E (Sects. 9.1, 9.2, 9.3) ε (Sect. 11) α (Sect. 12)

Figure 4: Four steps for calibrating and correcting the standard measurements for systematic errors by means of the model equations and additional calibration measurements. On the left side the measurements and their Müller matrix representations are listed and on the right side the scalar equations and their parameters, which can be retrieved from our model and from additional calibration measurements.

The most general equation Eq. (86) for our lidar model, with e.g. a calibrator before the PBS, contains eight optical parameters of the four optical elements and the atmosphere, and four variables, i.e. the rotation angles of the optical elements and of the laser polarisation. Note, because detectors only detect the flux of light, the retardation of the polarising beam-splitter Δ_S is irrelevant. For each setup we first derive the general formulations (Eq. (86)). Then, in

order to reduce the complexity of the equations and to carve out the most important and useful relations, we neglect certain parameters and variables in the detailed equations of special cases. We often omit the explicit description of the laser emitter optics \mathbf{M}_E (Eq. (87)), which means that we assume the light emitted to the atmosphere as arbitrarily polarised (see App. E.2) $\mathbf{I}_E = \mathbf{M}_E \mathbf{I}_L = T_E \mathbf{I}_L |i_E \ q_E \ u_E \ v_E\rangle$. If necessary \mathbf{I}_E can be expanded in the final equations by the appropriate ones in App. E. But we also consider the more simple case of a rotated linearly polarised laser $\mathbf{I}_E = \mathbf{I}_L = I_L |1 \ c_{2\alpha} \ s_{2\alpha} \ 0\rangle$. Furthermore, it is quite easy to remove the cross talk of the polarising beam-splitter \mathbf{M}_S by means of additional polarisation filters behind it, which removes many terms in the equation (Eq. (88)). We call such an analyser “cleaned”. The rotation γ of the receiving optics \mathbf{M}_O is very disturbing, which can be avoided in the very beginning of the lidar design (Eq. (89)). And at last, this paper provides the tools to determine how good a calibrator must be to be considered as ideal. With such a calibrator the equations become less complex (Eq.(90)).

$$\mathbf{I}_S = \eta_S \mathbf{M}_S(D_S) \mathbf{R}_y \mathbf{C}(D_C, \Delta_C, \varepsilon) \mathbf{M}_O(D_O, \Delta_O, \gamma) \mathbf{F}(a) \mathbf{M}_E(D_E, \Delta_E, \beta) \mathbf{I}_L(\alpha) \quad (86)$$

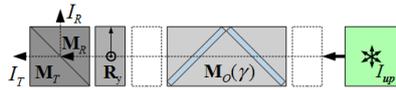
$$\mathbf{I}_S = \eta_S \mathbf{M}_S(D_S) \mathbf{R}_y \mathbf{C}(D_C, \Delta_C, \varepsilon) \mathbf{M}_O(D_O, \Delta_O, \gamma) \mathbf{F}(a) \quad \mathbf{I}_E \quad (87)$$

$$\mathbf{I}_S = \eta_S \mathbf{M}_{S_{clean}} \mathbf{R}_y \mathbf{C}(D_C, \Delta_C, \varepsilon) \mathbf{M}_O(D_O, \Delta_O, \gamma) \mathbf{F}(a) \quad \mathbf{I}_E \quad (88)$$

$$\mathbf{I}_S = \eta_S \mathbf{M}_S(D_S) \mathbf{R}_y \mathbf{C}(D_C, \Delta_C, \varepsilon) \mathbf{M}_O(D_O, \Delta_O, 0) \mathbf{F}(a) \quad \mathbf{I}_E \quad (89)$$

$$\mathbf{I}_S = \eta_S \mathbf{M}_S(D_S) \mathbf{R}_y \quad \mathbf{C}_{ideal} \quad \mathbf{M}_O(D_O, \Delta_O, \gamma) \mathbf{F}(a) \quad \mathbf{I}_E \quad (90)$$

6 Calibration with unpolarised input before the receiving optics



In principle, an additional light source with a known state of polarisation, which is placed before the telescope, can be used for the calibration. For other states of polarisation of the calibration light source the equations in Sect. (7.2) can be used together with the appropriate description of the input Stokes vector. But the beam from an additional light source has some disadvantages, because it fills the apertures of the individual optical elements differently than the backscattered light from the lidar laser, and also the distribution of the incident angles on elements with limited acceptance angles, as dichroic beams splitters and interference filters, is different. Furthermore, the wavelength band of the light source is usually different from that of the lidar laser, which introduces wavelength dependent transmission, diattenuation, and retardation effects. This can lead to errors in the calibration factor, which can additionally be range dependent. Such errors are very difficult to assess. We therefore prefer to use the atmospheric backscatter of the lidar laser for the calibration, which provides the same spatial and angular characteristics and the same wavelengths for the calibration as for the measurements. Nevertheless, the output Stokes vector \mathbf{I}_S of an unpolarised light source before the receiving optics is given by Eq. (91).

$$\mathbf{I}_S = \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O \mathbf{I}_{up} \Rightarrow \mathbf{A}_S = \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O \quad \text{and} \quad \mathbf{I}_{in} = \mathbf{I}_{up} \quad (91)$$

With the analyser vector from Eq. (D.7) and the unpolarised input Stokes vector \mathbf{I}_{in} before the lidar optics from Eq. (92) we get the calibration signals in Eq. (93).

$$\mathbf{I}_{in} = \mathbf{I}_{up} = I_{up} |1 \ 0 \ 0 \ 0\rangle \quad (92)$$

$$I_S = \eta_S \langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) | I_{up} = \eta_S T_S T_O I_{up} (1 + y D_S c_{2\gamma} D_O) \quad (93)$$

The gain ratio can be retrieved directly with Eq. (93)

$$\eta^* = \frac{I_R}{I_T} = \frac{\eta_R T_R}{\eta_T T_T} \frac{1 + y D_R D_O c_{2\gamma}}{1 + y D_T D_O c_{2\gamma}} = \eta \frac{1 + y D_R D_O c_{2\gamma}}{1 + y D_T D_O c_{2\gamma}} \quad (94)$$

Error sources are the unknown receiver optics rotation γ and the diattenuation D_O . With a cleaned analyser \mathbf{M}_S (see Suppl. S.10.10) and $\gamma = 0$ we get from Eq. (94)

$$\frac{\eta^*}{\eta} = \frac{1 - y D_O}{1 + y D_O} \quad (95)$$

With $D_O = \frac{T_O^p - T_O^s}{T_O^p + T_O^s}$ we get the gain ratios for the two setups $y = \pm 1$ from

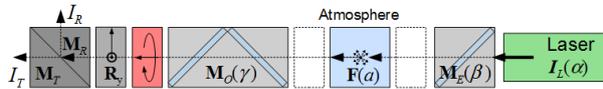
$$\frac{\eta^*(y = +1)}{\eta} = \frac{T_O^s}{T_O^p}, \quad \frac{\eta^*(y = -1)}{\eta} = \frac{T_O^p}{T_O^s} \quad (96)$$

As there are no calibrator induced rotational errors ε , all equations for the standard measurements of Sect. 4 are with $\varepsilon = 0^\circ$.

7 Calibration with a rotator - mechanical or by $\lambda/2$ plate (HWP)

With an ideal HWP rotator the input Stokes vector is rotated with respect to the coordinate system, while with the mechanical rotator the polarising beam-splitter and, if so, the receiving optics are rotated in the opposite direction to achieve the same effect. Mathematically the latter means a rotation of the coordinate system (see S.5). Furthermore, the rotation with a HWP includes a retardance of 180° and hence a mirroring of the input Stokes vector (see Eq. (S.10.13.2)). We combine the two methods in the rotator matrix \mathbf{M}_{rot} (Suppl. S.10.15) by introducing the rotator operator h (Eq. (S.10.15.1)), which is $h = +1$ for the mechanical rotator and $h = -1$ for the HWP rotator.

7.1 Calibration with a rotator before the polarising beam-splitter



The general formula for the output Stokes vector I_S with a rotation calibrator \mathbf{M}_{rot} before the polarising beam-splitter is Eq. (97).

$$\begin{aligned} I_S &= \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_{rot}(x45^\circ + \varepsilon, h) \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) I_L(\alpha) = \\ &= \eta_S \mathbf{A}_S(y) \mathbf{M}_{rot}(x45^\circ + \varepsilon, h) I_{in}(\gamma, a, \beta, \alpha) \end{aligned} \quad (97)$$

With the analyser part \mathbf{A}_S from Eq. (D.5), \mathbf{M}_{rot} from Eq. (S.10.15.1), and the input Stokes vector I_{in} from App. E.4 we get Eq. (98) for the calibration signals, and with the expanded input Stokes vector Eq. (E.31) we get from Eq. (98) the general calibration signals Eq. (99).

$$\begin{aligned}
\frac{I_S}{\eta_S T_S T_{rot} I_{in}} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{M}_{rot}(x45^\circ + \varepsilon, h) | \mathbf{I}_{in} \rangle}{T_S T_{rot} I_{in}} = \\
&= \left\langle \begin{array}{c} 1 \\ yD_S \\ 0 \\ 0 \end{array} \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -xs_{2\varepsilon} & -xhc_{2\varepsilon} & 0 \\ 0 & xc_{2\varepsilon} & -xhs_{2\varepsilon} & 0 \\ 0 & 0 & 0 & h \end{array} \right| \begin{array}{c} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{array} \right\rangle = \left\langle \begin{array}{c} 1 \\ -xys_{2\varepsilon} D_S \\ -xyhc_{2\varepsilon} D_S \\ 0 \end{array} \left| \begin{array}{c} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{array} \right\rangle = \\
&= i_{in} - xyD_S (s_{2\varepsilon} q_{in} + hc_{2\varepsilon} u_{in})
\end{aligned} \tag{98}$$

$$\begin{aligned}
\frac{I_S}{\eta_S T_S T_{rot} T_O F_{11} T_E I_L} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{M}_{rot}(x45^\circ + \varepsilon, h) | \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{I}_E \rangle}{T_S T_{rot} T_O F_{11} T_E I_L} = \\
&= i_E + aD_O (c_{2\gamma} q_E - s_{2\gamma} u_E) - xyD_S \left\{ \begin{array}{l} s_{2\varepsilon+h2\gamma} D_O i_E + a(s_{2\varepsilon} q_E - hc_{2\varepsilon} u_E) + \\ + hc_{2\varepsilon+h2\gamma} [W_O a (s_{2\gamma} q_E + c_{2\gamma} u_E) + Z_O s_O (1-2a) v_E] \end{array} \right\}
\end{aligned} \tag{99}$$

Since i_{in} in Eq. (98) is independent of ε , x , and y , we can define the function E in Eq. (100) and get for the calibration signals Eq. (101) and for the gain ratios η^* (Sect. 5) Eq. (102).

$$\begin{aligned}
E(\varepsilon, h, \gamma, a, \beta, \alpha) &\equiv \frac{s_{2\varepsilon} q_{in} + hc_{2\varepsilon} u_{in}}{i_{in}} = \\
&= \frac{s_{2\varepsilon+h2\gamma} D_O i_E + a(s_{2\varepsilon} q_E - hc_{2\varepsilon} u_E) + hc_{2\varepsilon+h2\gamma} [W_O a (s_{2\gamma} q_E + c_{2\gamma} u_E) + Z_O s_O (1-2a) v_E]}{i_E + aD_O (c_{2\gamma} q_E - s_{2\gamma} u_E)}
\end{aligned} \tag{100}$$

$$I_S = \eta_S T_S T_{rot} I_{in} [1 - xyD_S E] i_{in} \tag{101}$$

$$\eta^* = \frac{I_R}{I_T} = \frac{\eta_R T_R}{\eta_T T_T} \frac{1 - xyD_R E}{1 - xyD_T E} = \eta \frac{1 - xyD_R E}{1 - xyD_T E} \tag{102}$$

Eq. (103) shows the gain ratio from the $\Delta 90$ -calibration, assuming that the polarisation parameter a doesn't change in the calibration range between the two calibration measurements, i.e. $E_+ = E_-$ (see Sect. 5).

$$\frac{\eta_{\Delta 90}^*}{\eta} = \sqrt{\frac{1 - yD_R E_+}{1 - yD_T E_+} \frac{1 + yD_R E_-}{1 + yD_T E_-}} = \sqrt{\frac{1 - D_R^2 E^2}{1 - D_T^2 E^2}} \tag{103}$$

• **Special cases:** We immediately see that it is advantageous to use a cleaned analyser (see Suppl. S.10.10), because with $D_T = 1$, $D_R = -1$ Eq. (102) becomes Eq. (104) and all possible errors in the $\Delta 90$ -calibration from Eq. (103) are removed in Eq. (105), besides the problem of temporal change of a .

$$\begin{aligned}
D_T = +1, D_R = -1 &\Rightarrow \\
\frac{\eta^*}{\eta} &= \frac{1 + xyE}{1 - xyE}
\end{aligned} \tag{104}$$

$$\frac{\eta_{\Delta 90}^*}{\eta} = \sqrt{\frac{1 - E^2}{1 - E^2}} = 1 \Rightarrow \eta = \eta_{\Delta 90}^* \tag{105}$$

- From Eq. (100) we get Eq. (106) without emitter and receiver optics rotation, without laser rotation, but with calibrator rotation ε and with a horizontal-linearly polarised laser \mathbf{I}_L (Eq. (E.5)).

$$\gamma = \beta = \alpha = 0 \wedge \mathbf{I}_E = \mathbf{I}_L = I_L |1 \ 1 \ 0 \ 0\rangle \Rightarrow$$

$$E(\varepsilon, h, 0, a, 0, 0) = s_{2\varepsilon} \frac{D_O + a}{1 + aD_O} \quad (106)$$

- If additionally without calibrator rotation error ε , Eq. (106) becomes Eq. (107) and thus η^* and $\eta^*_{\Delta 90}$ are independent of the atmospheric polarisation parameter a and any atmospheric changes (see Eqs. (102) and (103)).

$$\varepsilon = 0 \Rightarrow$$

$$E(0, h, 0, a, 0, 0) = 0 \quad (107)$$

- A more general case without receiver optics rotation γ and without calibrator rotation ε , but with unknown laser and emitter optics rotation, Eq. (100) becomes Eq. (108).

$$\text{with } \gamma = \varepsilon = 0 \Rightarrow E(0, h, 0, a, \beta, \alpha) = \frac{u_m}{i_{in}} = \frac{hZ_O [s_O(1-2a)v_E - c_O a u_E] + (h-1) a u_E}{i_E + aD_O q_E} \quad (108)$$

Eq. (108) stays quite complex if we use \mathbf{I}_E with rotated emitter optics (Eq. (E.12)), and even if we assume a linearly polarised laser (Eq. (E.9)).

- With a horizontal-linearly polarised laser (Eq. (E.13)) aligned with the rotated emitter optics ($\alpha = \beta$) we get from Eq. (100)

$$\text{with } \alpha = \beta \wedge \mathbf{I}_E = T_E I_L (1 + D_E) |1 \ c_{2\alpha} \ s_{2\alpha} \ 0\rangle \Rightarrow$$

$$E(\varepsilon, h, \gamma, a, \alpha, \alpha) = \frac{s_{2\varepsilon+h2\gamma} D_O + a \left[(s_{2\varepsilon} c_{2\alpha} - hc_{2\varepsilon} s_{2\alpha}) + hc_{2\varepsilon+h2\gamma} W_O (s_{2\gamma} c_{2\alpha} + c_{2\gamma} s_{2\alpha}) \right]}{1 + aD_O (c_{2\gamma} c_{2\alpha} - s_{2\gamma} s_{2\alpha})} =$$

$$= \frac{s_{2\varepsilon+h2\gamma} D_O + a (hc_{2\varepsilon+h2\gamma} s_{2\gamma+2\alpha} W_O + s_{2\varepsilon-h2\alpha})}{1 + aD_O c_{2\gamma+2\alpha}} \quad (109)$$

Note: $D_E = 0$ means without emitter optics, and $W_O = (1 - Z_O c_O)$.

- Eq. (109) with laser, emitter and receiver optics aligned with each other becomes

$$\text{with } \alpha = \beta = -\gamma \wedge \mathbf{I}_E = T_E I_L (1 + D_E) |1 \ c_{2\alpha} \ s_{2\alpha} \ 0\rangle \Rightarrow$$

$$E(\varepsilon, h, \gamma, a, -\gamma, -\gamma) = s_{2\varepsilon+h2\gamma} \frac{D_O + ha}{1 + aD_O} \quad (110)$$

- Eq. (109) with receiver optics and calibrator aligned \Rightarrow

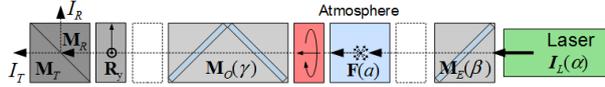
$$\text{with } \alpha = \beta, \varepsilon = -h\gamma \wedge \mathbf{I}_E = T_E I_L (1 + D_E) |1 \ c_{2\alpha} \ s_{2\alpha} \ 0\rangle \Rightarrow$$

$$E(-\gamma, h, \gamma, a, \alpha, \alpha) = \frac{hs_{2\gamma+2\alpha} a (1 - Z_O c_O - h)}{1 + aD_O c_{2\gamma+2\alpha}} \quad (111)$$

In summary: the $\Delta 90$ -calibration with a cleaned analyser results in a calibration factor η independent of \mathbf{I}_{in} , i.e independent of any optics before the calibrator and independent of the rotation error ε of the calibrator. Calibrations without a cleaned analyser include error terms

which increase rapidly with increasing ε and α for the individual $\pm 45^\circ$ calibrations (Bravo-Aranda et al. 2016), because D_T and D_R in the numerator and denominator have opposite signs in Eq. (102). The geometric mean of the two $\pm 45^\circ$ calibrations in Eq. (103) removes the opposite signs and the increasing error with increasing ε and α is reduced by orders of magnitude compared to the individual $\pm 45^\circ$ calibrations (Freudenthaler et al. 2009).

7.2 Calibration with a rotator before the receiving optics



The general formula for the output Stokes vector I_S with rotation calibrator before the receiving optics \mathbf{M}_O and the polarising beam-splitter \mathbf{M}_S is given in Eq. (112).

$$\begin{aligned} I_S &= \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{M}_{rot}(x45^\circ + \varepsilon, h) \mathbf{F}(a) \mathbf{M}_E(\beta) I_L(\alpha) = \\ &= \eta_S \mathbf{A}_S(y, \gamma) \mathbf{M}_{rot}(x45^\circ + \varepsilon, h) I_{in}(a, \beta, \alpha) \end{aligned} \quad (112)$$

With \mathbf{A}_S from Eq. (D.7), \mathbf{M}_{rot} from Eq. (S.10.15.1), and I_{in} from App. E.3, i.e. Eq. (E.19), we get Eq. (113) for the calibration signals using the trigonometric relations in Suppl. S.12.

$$\begin{aligned} \frac{I_S}{\eta_S T_S T_O T_{rot} F_{11} T_E I_L} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) | \mathbf{M}_{rot}(x45^\circ + \varepsilon, h) | \mathbf{F}(a) I_E \rangle}{T_S T_O T_{rot} F_{11} T_E I_L} = \\ &= \left\langle \begin{array}{c} 1 + y c_{2\gamma} D_O D_S \\ c_{2\gamma} D_O + y D_S (1 - s_{2\gamma}^2 W_O) \\ s_{2\gamma} (D_O + y c_{2\gamma} D_S W_O) \\ -y s_{2\gamma} D_S Z_O S_O \end{array} \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -x s_{2\varepsilon} & -x h c_{2\varepsilon} & 0 \\ 0 & x c_{2\varepsilon} & -x h s_{2\varepsilon} & 0 \\ 0 & 0 & 0 & h \end{array} \right| \begin{array}{c} i_E \\ a q_E \\ -a u_E \\ (1-2a) v_E \end{array} \right\rangle = \\ &= (1 + y c_{2\gamma} D_O D_S) i_E - y h s_{2\gamma} D_S Z_O S_O (1-2a) v_E + \\ &- x a \left\{ D_O (q_E s_{2\varepsilon-2\gamma} - h u_E c_{2\varepsilon-2\gamma}) - y D_S [s_{2\gamma} W_O (q_E c_{2\varepsilon-2\gamma} + h u_E s_{2\varepsilon-2\gamma}) - (q_E s_{2\varepsilon} - h u_E c_{2\varepsilon})] \right\} \end{aligned} \quad (113)$$

Special cases: Without receiver optics rotation, i.e. $\gamma = 0$, Eq. (113) becomes Eq. (114), which is less complex and independent of retardation terms $Z_O S_O$ and W_O , and the gain ratios η^* (Sect. 5) can be written as Eqs. (115) and (116).

$$\gamma = 0 \Rightarrow \frac{I_S}{(\eta_S T_S T_O T_{rot} F_{11} T_E I_L)} = (1 + y D_O D_S) i_E - x a (D_O + y D_S) (q_E s_{2\varepsilon} - h u_E c_{2\varepsilon}) \quad (114)$$

$$\frac{\eta^*}{\eta} = \frac{(1 + y D_O D_R) i_E - x a (D_O + y D_R) (q_E s_{2\varepsilon} - h u_E c_{2\varepsilon})}{(1 + y D_O D_T) i_E - x a (D_O + y D_T) (q_E s_{2\varepsilon} - h u_E c_{2\varepsilon})} \quad (115)$$

$$\frac{\eta_{\Delta 90}^*}{\eta} = \frac{\sqrt{(1 + y D_O D_R)^2 i_E^2 - a^2 (D_O + y D_R)^2 (q_E s_{2\varepsilon} - h u_E c_{2\varepsilon})^2}}{\sqrt{(1 + y D_O D_T)^2 i_E^2 - a^2 (D_O + y D_T)^2 (q_E s_{2\varepsilon} - h u_E c_{2\varepsilon})^2}} \quad (116)$$

• With a cleaned analyser (see Suppl. S.10.10) Eqs. (115) and (116) become Eqs. (117) and (118).

$$\gamma = 0^\circ, D_T = +1, D_R = -1 \Rightarrow$$

$$\frac{\eta^*}{\eta} = \frac{1 - yD_O i_E + xy a (q_E s_{2\varepsilon} - hu_E c_{2\varepsilon})}{1 + yD_O i_E - xy a (q_E s_{2\varepsilon} - hu_E c_{2\varepsilon})} \quad (117)$$

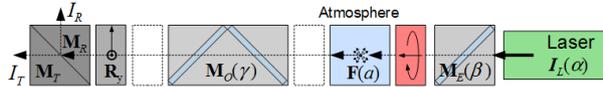
$$\frac{\eta_{\Delta 90}^*}{\eta} = \frac{1 - yD_O}{1 + yD_O} \quad (118)$$

The gain ratio $\eta_{\Delta 90}^*$ in Eq. (118) is independent of the input Stokes vector, i.e. the laser polarisation, independent of the calibrator type (mechanical or $\lambda/2$ plate rotation) and of the calibrator rotation ε . Using the two calibration setups Eqs. (118) (before the receiving optics) and (105) (before the PBS) it is possible to retrieve the receiver optics diattenuation parameter D_O . Furthermore, with this setup and the measured gain ratio $\eta_{\Delta 90}^*$ from Eq. (118) we get the polarisation parameter a (Eq.(119)) and the backscatter coefficient F_{11} (Eq.(120)) with Eq. (78) directly from the measurement signals I_R and I_T according to Eqs. (61) and (65) without the explicit knowledge of D_O or any other correction.

$$a = y \frac{\eta_{\Delta 90}^* I_T - I_R}{\eta_{\Delta 90}^* I_T + I_R} \quad (119)$$

$$F_{11} \propto \eta_{\Delta 90}^* I_T + I_R \quad (120)$$

7.3 Calibration with a rotator behind the emitter optics



The general formula for the output Stokes vector I_S with rotation calibrator \mathbf{M}_{rot} (Eq. (S.10.15.2)) behind the emitter optic \mathbf{M}_E and all derivations therefrom can be derived from the previous Sect. 7.2 using Eq. (121) and considering the mirror effect of \mathbf{F} and the associated sign changes in the rotation angle (Suppl. S.6.3) when mathematically moving the calibrator \mathbf{M}_{rot} from behind the emitter optics \mathbf{M}_E to before the receiving optics \mathbf{M}_O . Regarding the rotation and mirror relations see Suppl. S.5 and S.6.

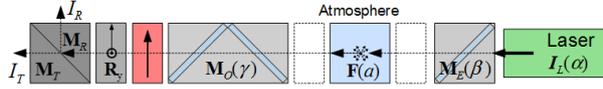
$$\begin{aligned} I_S &= \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_{rot}(x45^\circ + \varepsilon, h) \mathbf{M}_E(\beta) I_L(\alpha) = \\ &= \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{R}(x45^\circ) \mathbf{R}(\varepsilon) \mathbf{M}_h \mathbf{M}_E(\beta) I_L(\alpha) = \\ &= \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{R}(-x45^\circ) \mathbf{R}(-\varepsilon) \mathbf{M}_h \mathbf{F}(a) \mathbf{M}_E(\beta) I_L(\alpha) \\ &= \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{M}_{rot}(-x45^\circ - \varepsilon, h) \mathbf{F}(a) \mathbf{M}_E(\beta) I_L(\alpha) \end{aligned} \quad (121)$$

8 Calibration with a linear polariser (P)

A linear polariser is a retarding linear diattenuator (Suppl. . S.10.3). The output of an ideal linear polariser is linearly polarised light independent of the state of polarisation of the input, which seems to be ideal for our purpose. Polarising sheet filters are thin and have large acceptance angles. Hence they can be easily included in existing lidar systems, even in diverging or converging light paths as close to the telescope focus. However, to achieve an acceptable uncertainty of the calibration factor, a rather good extinction ratio of the linear polariser of order 10^{-4} and better is necessary. Crystal polarisers exhibit such high extinction ratios, but the available diameters are limited, they are bulky and have smaller acceptance angles. Wire grid and liquid crystal polarisers usually don't show high enough extinction

ratios. A linear polariser is described in the same way as a polarising beam-splitter, which is a retarding diattenuator (Suppl. S.4 and S.10.3 ff), with high diattenuation ($\mathbf{D}_P \approx 1$). Since the standard measurements have to be performed without the linearly polarising calibrator, there is no rotational misalignment ε for the standard measurement signals of Sect. 4. As the equations become too complex with a real linear polariser with diattenuation and retardation, we use a real linear polariser only in Sect. 8.1 to show as an example how the uncertainty of the extinction ratio influences the accuracy of the calibration factor, and else we use an ideal linear polariser. The general formula with a real linear polariser can be found in App. C.2.

8.1 Calibration with a linear polariser before the polarising beam-splitter



$$\begin{aligned} I_S &= \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_P(x45^\circ + \varepsilon) \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) I_L(\alpha) = \\ &= \eta_S \mathbf{A}_S(y) \mathbf{M}_P(x45^\circ + \varepsilon) I_{in}(\gamma, a, \beta, \alpha) \end{aligned} \quad (122)$$

With Eq. (D.5) for the analyser part \mathbf{A}_S , Eq. (S.10.6.1) for the rotated linear polariser, and I_{in} from App. E.4 we get the general calibration signals Eq. (123).

$$\begin{aligned} \frac{I_S}{\eta_S T_S T_P I_{in}} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{M}_P(x45^\circ + \varepsilon) | I_{in} \rangle}{T_S T_P I_{in}} = \\ &= \left\langle \begin{array}{c} 1 - x y s_{2\varepsilon} D_P D_S \\ -x s_{2\varepsilon} D_P + y D_S (1 - c_{2\varepsilon}^2 W_P) \\ x c_{2\varepsilon} D_P - y s_{2\varepsilon} c_{2\varepsilon} W_P D_S \\ -x y c_{2\varepsilon} Z_P s_P D_S \end{array} \middle| \begin{array}{c} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{array} \right\rangle = \\ &= i_{in} + y D_S [q_{in} - c_{2\varepsilon} W_P (c_{2\varepsilon} q_{in} + s_{2\varepsilon} u_{in})] - \\ &- x [D_P (s_{2\varepsilon} q_{in} - c_{2\varepsilon} u_{in}) + y D_S (s_{2\varepsilon} D_P i_{in} + c_{2\varepsilon} Z_P s_P v_{in})] \end{aligned} \quad (123)$$

• Special cases: Without calibrator rotation error ε Eq.(123) becomes Eq. (124).

$$\varepsilon = 0 \Rightarrow$$

$$\begin{aligned} \frac{I_S}{\eta_S T_S T_P I_{in}} &= i_{in} + y D_S [1 - W_P] q_{in} + x [u_{in} D_P - y D_S Z_P s_P v_{in}] = \\ &= i_{in} + x D_P u_{in} + y D_S Z_P (c_P q_{in} - s_P v_{in}) \end{aligned} \quad (124)$$

• We get with a cleaned analyser and horizontal-linearly polarised input I_{in} , with Eq. (124) the gain ratios (Sect. 5) in Eq. (125).

$$\varepsilon = 0, D_T = +1, D_R = -1, I_{in} = |1 \ 1 \ 0 \ 0\rangle \Rightarrow$$

$$\frac{\eta^*}{\eta} = \frac{1 - y Z_P c_P}{1 + y Z_P c_P} \quad (125)$$

• Using Eq. (S.10.10.8) for the extinction ratio ρ of the real linear polariser, we get the approximation Eq. (126) for the gain ratios depending on ρ , with which we can estimate the error of the gain ratio if we use a real polariser with extinction ratio ρ for the measurements

but assume an ideal polariser as calibrator in the correction equations. Eq. (126) with $\rho = 10^{-5}$ and $\rho = 10^{-4}$, e.g., gives relative errors of the gain ratios of about 1.3% and 8%, respectively.

$$\text{with } \rho = k_2/k_1 \text{ and } k_2 \ll k_1 \Rightarrow \frac{\eta^*}{\eta} \approx \frac{1 - 2y\sqrt{\rho c_P}}{1 + 2y\sqrt{\rho c_P}} \approx 1 - 4y\sqrt{\rho c_P} \quad (126)$$

• With an ideal linear polariser Eq.(123) becomes Eq. (127), and the gain ratios Eq. (128) are independent of I_{in} , i.e. independent of the laser polarisation, of the atmospheric depolarisation, and of any optics before the calibrator. The error due to the calibrator rotation ε is largely reduced with the $\Delta 90$ -calibration in Eq. (129) compared to the $\pm 45^\circ$ -calibration in Eq. (128).

$$D_P = 1 \Rightarrow W_P = 1, Z_P = 0 \Rightarrow \frac{I_S}{\eta_S T_S T_P I_{in}} = (1 - x y s_{2\varepsilon} D_S) \langle 1 \quad -x s_{2\varepsilon} \quad x c_{2\varepsilon} \quad 0 | i_{in} \quad q_{in} \quad u_{in} \quad v_{in} \rangle = \quad (127)$$

$$= (1 - x y s_{2\varepsilon} D_S) [i_{in} - x (s_{2\varepsilon} q_{in} - c_{2\varepsilon} u_{in})] \frac{\eta^*}{\eta} = \frac{1 - x y s_{2\varepsilon} D_R}{1 - x y s_{2\varepsilon} D_T} \quad (128)$$

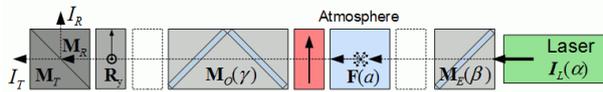
$$\frac{\eta_{\Delta 90}^*}{\eta} = \sqrt{\frac{1 - s_{2\varepsilon}^2 D_R^2}{1 - s_{2\varepsilon}^2 D_T^2}} \quad (129)$$

• If additionally a cleaned analyser is used (see Suppl. S.10.10), Eqs. (128) and (129) become Eqs. (130) and (131). Eq. (130) is of the form of Eq. (193) and can be used to determine ε (see Sect. 11). Eq. (131) shows that the $\Delta 90$ -calibration with a cleaned analyser is free of ε error.

$$\text{with } D_P = 1, W_P = 1, Z_P = 0, D_T = +1, D_R = -1 \Rightarrow \frac{\eta^*}{\eta} = \frac{1 + x y s_{2\varepsilon}}{1 - x y s_{2\varepsilon}} \quad (130)$$

$$\eta_{\Delta 90}^* = \eta = \frac{\eta_R T_R^S}{\eta_T T_T^P} \quad (131)$$

8.2 Calibration with an ideal linear polariser before the receiving optics



$$I_S = \eta_S \mathbf{M}_S \mathbf{R}(y) \mathbf{M}_O(\gamma) \mathbf{M}_P(x45^\circ + \varepsilon) \mathbf{F}(a) \mathbf{M}_E(\beta) I_L(\alpha) = \eta_S \mathbf{A}_S(y, \gamma) \mathbf{M}_P(x45^\circ + \varepsilon) I_{in}(a, \beta, \alpha) \quad (132)$$

With Eq. (D.7) for the analyser part \mathbf{A}_S , Eq. (S.10.8.6) for the ideal linear polariser \mathbf{M}_P , and any of the input Stokes vectors I_{in} of App. E.3 we get Eq. (133) for the calibration signals I_S . Since the last term of Eq. (133) is independent of the analyser diattenuation parameters D_S , this term cancels in the ratio of the gain ratios (Sect. 5) in Eq. (134), which are therefore independent of the input Stokes vector.

with $D_p = 1 \Rightarrow$

$$\frac{I_S}{\eta_S T_S T_O T_P F_{11} T_E I_L} = \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) | \mathbf{M}_P(x45^\circ + \varepsilon) | \mathbf{F}(a) \mathbf{I}_E \rangle}{T_S T_O T_P F_{11} T_E I_L} =$$

$$= \left\langle \begin{array}{c} 1 + y c_{2\gamma} D_S D_O \\ c_{2\gamma} D_O + y D_S (1 - s_{2\gamma}^2 W_O) \\ s_{2\gamma} (D_O + y c_{2\gamma} D_S W_O) \\ -y s_{2\gamma} D_S Z_O S_O \end{array} \left| \begin{array}{c} 1 \\ -x s_{2\varepsilon} \\ x c_{2\varepsilon} \\ 0 \end{array} \right. \left\langle \begin{array}{c} 1 \\ -x s_{2\varepsilon} \\ x c_{2\varepsilon} \\ 0 \end{array} \left| \begin{array}{c} i_E \\ a q_E \\ -a u_E \\ (1 - 2a) v_E \end{array} \right. \right\rangle =$$

$$= \left[(1 + y c_{2\gamma} D_S D_O) - x \left\{ s_{2\varepsilon - 2\gamma} D_O + y D_S [s_{2\varepsilon} - s_{2\gamma} c_{2\varepsilon - 2\gamma} W_O] \right\} \right] [i_E - x a (s_{2\varepsilon} q_E + c_{2\varepsilon} u_E)]$$

$$\frac{\eta^*}{\eta} = \frac{(1 + y c_{2\gamma} D_O D_R) - x [s_{2\varepsilon - 2\gamma} D_O + y D_R (s_{2\varepsilon} - s_{2\gamma} c_{2\varepsilon - 2\gamma} W_O)]}{(1 + y c_{2\gamma} D_O D_T) - x [s_{2\varepsilon - 2\gamma} D_O + y D_T (s_{2\varepsilon} - s_{2\gamma} c_{2\varepsilon - 2\gamma} W_O)]} \quad (134)$$

• Special cases: Eq.(134) gets neither • with a cleaned analyser alone (Eq. (135)) nor • without receiver optics rotation γ alone (Eq. (136)) very simple, but • with both conditions Eq. (137) is of the form of Eq. (193) and can be used to estimate the calibrator rotation ε (see Sect. 11). The corresponding $\Delta 90$ -calibration in Eq. (138) can be used together with the calibration measurements which directly yield η (see Eqs. (131) or (105), for example) to determine the diattenuation parameter D_O of the receiving optics.

with $D_p = 1, D_T = +1, D_R = -1 \Rightarrow$

$$\frac{\eta^*}{\eta} = \frac{(1 - y c_{2\gamma} D_O) - x [s_{2\varepsilon - 2\gamma} D_O - y (s_{2\varepsilon} - s_{2\gamma} c_{2\varepsilon - 2\gamma} W_O)]}{(1 + y c_{2\gamma} D_O) - x [s_{2\varepsilon - 2\gamma} D_O + y (s_{2\varepsilon} - s_{2\gamma} c_{2\varepsilon - 2\gamma} W_O)]} \quad (135)$$

with $D_p = 1, \gamma = 0 \Rightarrow$

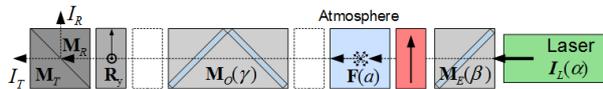
$$\frac{\eta^*}{\eta} = \frac{(1 + y D_O D_R) - x s_{2\varepsilon} [D_O + y D_R]}{(1 + y D_O D_T) - x s_{2\varepsilon} [D_O + y D_T]} \quad (136)$$

with $D_p = 1, D_T = +1, D_R = -1, \gamma = 0 \Rightarrow$

$$\frac{\eta^*}{\eta} = \frac{(1 - y D_O) - x s_{2\varepsilon} (D_O - y)}{(1 + y D_O) - x s_{2\varepsilon} (D_O + y)} = \frac{1 - y D_O}{1 + y D_O} \frac{1 + x y s_{2\varepsilon}}{1 - x y s_{2\varepsilon}} \quad (137)$$

$$\frac{\eta_{\Delta 90}^*}{\eta} = \frac{1 - y D_O}{1 + y D_O} \quad (138)$$

8.3 Calibration with an ideal linear polariser behind the emitter optics



$$I_S = \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_P(x45^\circ + \varepsilon) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha) =$$

$$= \eta_S \mathbf{A}_S(y, \gamma, a) \mathbf{M}_P(x45^\circ + \varepsilon) \mathbf{I}_{in}(\beta, \alpha) \quad (139)$$

With Eq. (D.13) for the analyser part \mathbf{A}_S , Eq. (S.10.8.6) for the ideal linear polariser \mathbf{M}_P , and any of the emitter Stokes vectors \mathbf{I}_E of App. E.2 we get the calibration signals I_S in Eq. (140). Since the the last term of Eq. (140) is independent of analyser diattenuation parameters D_S , it cancels in the ratio of the gain ratios (Sect. 5), and the gain ratios in Eq. (141) are independent of the input Stokes vector.

with $D_P = 1 \Rightarrow$

$$\frac{I_S}{\eta_S T_S T_O F_{11} T_P I_{in}} = \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) | \mathbf{F}(a) \mathbf{M}_P(x45^\circ + \varepsilon) | \mathbf{I}_E \rangle}{T_S T_O F_{11} T_P I_E} =$$

$$= \left\langle \begin{array}{c} 1 + yc_{2\gamma} D_S D_O \\ a [c_{2\gamma} D_O + yD_S (1 - s_{2\gamma}^2 W_O)] \\ -a s_{2\gamma} (D_O + yc_{2\gamma} D_S W_O) \\ -(1 - 2a) ys_{2\gamma} D_S Z_O S_O \end{array} \right| \left| \begin{array}{c} 1 \\ -xs_{2\varepsilon} \\ xc_{2\varepsilon} \\ 0 \end{array} \right\rangle \left\langle \begin{array}{c} 1 \\ -xs_{2\varepsilon} \\ xc_{2\varepsilon} \\ 0 \end{array} \right| \left| \begin{array}{c} i_E \\ q_E \\ u_E \\ v_E \end{array} \right\rangle = \quad (140)$$

$$= \left[(1 + yc_{2\gamma} D_S D_O) - ax \left\{ s_{2\varepsilon+2\gamma} D_O + yD_S [s_{2\varepsilon} + s_{2\gamma} c_{2\varepsilon+2\gamma} W_O] \right\} \right] [i_E - x(s_{2\varepsilon} q_E - c_{2\varepsilon} u_E)]$$

$$\frac{\eta^*}{\eta} = \frac{(1 + yc_{2\gamma} D_O D_R) - xa [s_{2\varepsilon+2\gamma} D_O + yD_R (s_{2\varepsilon} + s_{2\gamma} c_{2\varepsilon+2\gamma} W_O)]}{(1 + yc_{2\gamma} D_O D_T) - xa [s_{2\varepsilon+2\gamma} D_O + yD_T (s_{2\varepsilon} + s_{2\gamma} c_{2\varepsilon+2\gamma} W_O)]} \quad (141)$$

• **Special cases:** Eq.(141) with a cleaned analyser becomes Eq. (142), without receiver optics rotation Eq. (143), and with both conditions Eq. (144). Eq. (144) is of the form of Eq. (199) and can be used to determine ε (see Sect. 11). As before in Eq.(138) the corresponding $\Delta 90$ -calibration becomes Eq. (145),

with $D_P = 1, D_T = +1, D_R = -1 \Rightarrow$

$$\frac{\eta^*}{\eta} = \frac{(1 - yc_{2\gamma} D_O) - xa [s_{2\varepsilon+2\gamma} D_O - y(s_{2\varepsilon} + s_{2\gamma} c_{2\varepsilon+2\gamma} W_O)]}{(1 + yc_{2\gamma} D_O) - xa [s_{2\varepsilon+2\gamma} D_O + y(s_{2\varepsilon} + s_{2\gamma} c_{2\varepsilon+2\gamma} W_O)]} \quad (142)$$

with $D_P = 1, \gamma = 0 \Rightarrow$

$$\frac{\eta^*}{\eta} = \frac{(1 + yD_O D_R) - xas_{2\varepsilon} [D_O + yD_R]}{(1 + yD_O D_T) - xas_{2\varepsilon} [D_O + yD_T]} \quad (143)$$

with $D_P = 1, D_T = +1, D_R = -1, \gamma = 0 \Rightarrow$

$$\frac{\eta^*}{\eta} = \frac{(1 - yD_O) - xas_{2\varepsilon} (D_O - y)}{(1 + yD_O) - xas_{2\varepsilon} (D_O + y)} = \frac{(1 - yD_O)(1 + xyas_{2\varepsilon})}{(1 + yD_O)(1 - xyas_{2\varepsilon})} \quad (144)$$

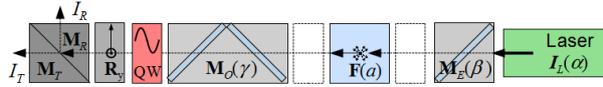
$$\frac{\eta_{\Delta 90}^*}{\eta} = \frac{1 - yD_O}{1 + yD_O} \quad (145)$$

9 Calibration with a $\lambda/4$ plate (QWP)

A $\lambda/4$ -plate (QWP) is a retarding linear diattenuator (S.4) with 90° phase shift between the polarisation parallel and perpendicular to the fast axis and without diattenuation (S.10.16 ff). Further details can be found in Bennett (2009a), Bennett (2009b), and Chipman (2009b). Oriented at $\pm 45^\circ$ relative to incident linear polarisation, its output is circularly polarised. Since the equations with a real QWP with retardation error ω (Suppl. S.10.16) are too complex, we

consider ω only in Sect. 9.1 to show with an example how this uncertainty influences the accuracy of the calibration factor. The general formula with a real QWP can be found in App. C.3.

9.1 Calibration with a $\lambda/4$ plate before the the polarising beam-splitter



$$\begin{aligned} I_S &= \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_{QW}(\times 45^\circ + \varepsilon, \omega) \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{I}_E = \\ &\eta_S \mathbf{A}_S(y) \mathbf{M}_{QW}(\times 45^\circ + \varepsilon, \omega) \mathbf{I}_{in}(\gamma, a, \beta, \alpha) \end{aligned} \quad (146)$$

With Eq. (D.5) for the analyser part \mathbf{A}_S , Eq. (S.10.16.3) for the $\lambda/4$ plate \mathbf{M}_{QW} with phase shift error ω , and with the input Stokes vector \mathbf{I}_{in} from App. E.4 we get the calibration signals I_S in Eq. (147).

$$\begin{aligned} \Delta_{QW} &= 90^\circ + \omega \Rightarrow \\ \frac{I_S}{\eta_S T_S T_{QW} I_{in}} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{M}_{QW}(\times 45^\circ + \varepsilon, \omega) | \mathbf{I}_{in} \rangle}{\eta_S T_S T_{QW} I_{in}} = \\ &= \left\langle \begin{array}{c} 1 \\ yD_S \\ 0 \\ 0 \end{array} \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & s_{2\varepsilon}^2 - c_{2\varepsilon}^2 s_\omega & -s_{2\varepsilon} c_{2\varepsilon} (1 + s_\omega) & -x c_{2\varepsilon} c_\omega \\ 0 & -s_{2\varepsilon} c_{2\varepsilon} (1 + s_\omega) & c_{2\varepsilon}^2 - s_{2\varepsilon}^2 s_\omega & -x s_{2\varepsilon} c_\omega \\ 0 & x c_{2\varepsilon} c_\omega & x s_{2\varepsilon} c_\omega & -s_\omega \end{array} \right| \begin{array}{c} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{array} \right\rangle = \\ &= i_{in} + yD_S [s_{2\varepsilon} (s_{2\varepsilon} q_{in} - c_{2\varepsilon} u_{in}) - c_{2\varepsilon} s_\omega (c_{2\varepsilon} q_{in} + s_{2\varepsilon} u_{in}) - x c_{2\varepsilon} c_\omega v_{in}] \end{aligned} \quad (147)$$

• **Special cases:** For the investigation of the effect of the phase shift error ω we neglect the rotation error ε in Eq. (147) and get the calibration signals Eq. (148) and the gain ratios Eq. (149).

$$\begin{aligned} \varepsilon &= 0 \Rightarrow \\ I_S &= \eta_S T_S T_{QW} I_{in} [i_{in} - yD_S (s_\omega q_{in} + x c_\omega v_{in})] \end{aligned} \quad (148)$$

$$\frac{\eta^*}{\eta} = \frac{i_{in} - yD_S s_\omega q_{in} - x y D_S c_\omega v_{in}}{i_{in} - yD_S s_\omega q_{in} - x y D_S c_\omega v_{in}} \quad (149)$$

• With a cleaned analyser, the gain ratios from Eq. (149) become Eq. (150) and for the $\Delta 90$ -calibration Eq. (151), from which we can estimate the influence of a phase shift error ω .

with $\varepsilon = 0, D_T = +1, D_R = -1 \Rightarrow$

$$\frac{\eta^*}{\eta} = \frac{i_{in} + y s_\omega q_{in} + x y c_\omega v_{in}}{i_{in} - y s_\omega q_{in} - x y c_\omega v_{in}} \quad (150)$$

$$\frac{\eta_{\Delta 90}^*}{\eta} = \sqrt{\frac{(i_{in} + y s_\omega q_{in})^2 - c_\omega^2 v_{in}^2}{(i_{in} - y s_\omega q_{in})^2 - c_\omega^2 v_{in}^2}} \quad (151)$$

• Without phase shift error ω in Eq. (147) but with calibrator rotation error ε we get the calibration signals Eq. (152) and the gain ratios Eq. (153).

$$\omega = 0 \Rightarrow$$

$$I_S = \eta_S T_S T_{QW} I_{in} \left\{ i_{in} + y D_S \left[s_{2\varepsilon} (s_{2\varepsilon} q_{in} - c_{2\varepsilon} u_{in}) - x c_{2\varepsilon} v_{in} \right] \right\} \quad (152)$$

$$\frac{\eta^*}{\eta} = \frac{i_{in} + y D_R s_{2\varepsilon} (s_{2\varepsilon} q_{in} - c_{2\varepsilon} u_{in}) - xy D_R c_{2\varepsilon} v_{in}}{i_{in} + y D_T s_{2\varepsilon} (s_{2\varepsilon} q_{in} - c_{2\varepsilon} u_{in}) - xy D_T c_{2\varepsilon} v_{in}} \quad (153)$$

$$\omega = 0, D_T = +1, D_R = -1 \Rightarrow$$

$$\frac{\eta_{\Delta 90}^*}{\eta} = \sqrt{\frac{(i_{in} - y s_{2\varepsilon} (s_{2\varepsilon} q_{in} - c_{2\varepsilon} u_{in}))^2 - c_{2\varepsilon}^2 v_{in}^2}{(i_{in} + y s_{2\varepsilon} (s_{2\varepsilon} q_{in} - c_{2\varepsilon} u_{in}))^2 - c_{2\varepsilon}^2 v_{in}^2}} \quad (154)$$

The terms without the x-factor in Eq. (150) containing ω and in Eq. (153) containing ε are not compensated with the $\Delta 90$ -calibration in Eq. (151) and Eq. (154), even if a cleaned analyser is used. This is a disadvantage of the QWP compared to the linear polariser (see Eq. (129)).

• From Eq.(153) without calibrator rotation ε we get the gain ratios Eqs. (155) and (156).

$$\omega = \varepsilon = 0 \Rightarrow$$

$$\frac{\eta^*}{\eta} = \frac{i_{in} - xy D_R v_{in}}{i_{in} - xy D_T v_{in}} \quad (155)$$

$$\frac{\eta_{\Delta 90}^*}{\eta} = \sqrt{\frac{i_{in}^2 - D_R^2 v_{in}^2}{i_{in}^2 - D_T^2 v_{in}^2}} \quad (156)$$

• With a cleaned analyser Eq. (156) becomes Eq. (157).

$$\omega = \varepsilon = 0, D_T = +1, D_R = -1 \Rightarrow$$

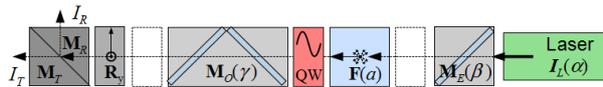
$$\eta_{\Delta 90}^* = \eta \quad (157)$$

• The advantage of the QWP calibrator is that we can retrieve from Eqs. (157) and (155) with a cleaned analyser the degree of circular polarisation v_{in}/i_{in} of the light before the polarising beam-splitter according to Eq. (158). Bear in mind that η^* and $\eta_{\Delta 90}^*$ in Eq. (158) are values directly derived from measured signals. The errors due to uncertainties in ε or ω can be estimated by means of equations earlier in this section.

$$\omega = \varepsilon = 0, D_T = +1, D_R = -1 \Rightarrow$$

$$\frac{v_{in}}{i_{in}} = \frac{1}{xy} \frac{\eta^* - \eta_{\Delta 90}^*}{\eta^* + \eta_{\Delta 90}^*} \quad (158)$$

9.2 Calibration with an ideal $\lambda/4$ plate before the receiving optics



$$I_S = \eta_S \mathbf{M}_S \mathbf{R}(y) \mathbf{M}_O(\gamma) \mathbf{M}_{QW}(x45^\circ + \varepsilon) \mathbf{F}(a) \mathbf{I}_E(\beta, \alpha) =$$

$$= \eta_S \mathbf{A}_S(y, \gamma) \mathbf{M}_{QW}(x45^\circ + \varepsilon) \mathbf{I}_{in}(a, \beta, \alpha) \quad (159)$$

With Eq. (D.7) for the analyser part \mathbf{A}_S , an ideal $\lambda/4$ plate \mathbf{M}_{QW} Eq. (S.10.17.3), and with an input Stokes vector \mathbf{I}_{in} from App. E.3 we get the general calibration signals I_S in Eq. (160).

$$\begin{aligned}
\frac{I_S}{\eta_S T_S T_O T_{QW} F_{11} T_E I_L} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) | \mathbf{M}_{QW}(x45^\circ + \varepsilon) | \mathbf{F}(a) \mathbf{M}_E I_L \rangle}{T_S T_O T_{QW} F_{11} T_E I_L} = \\
&= \left\langle \begin{array}{l} 1 + y c_{2\gamma} D_S D_O \\ c_{2\gamma} D_O + y D_S (1 - s_{2\gamma}^2 W_O) \\ s_{2\gamma} (D_O + y c_{2\gamma} D_S W_O) \\ -y s_{2\gamma} D_S Z_O S_O \end{array} \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & s_{2\varepsilon}^2 & -s_{2\varepsilon} c_{2\varepsilon} & -x c_{2\varepsilon} \\ 0 & -s_{2\varepsilon} c_{2\varepsilon} & c_{2\varepsilon}^2 & -x s_{2\varepsilon} \\ 0 & x c_{2\varepsilon} & x s_{2\varepsilon} & 0 \end{array} \right. \begin{array}{l} i_E \\ a q_E \\ -a u_E \\ (1-2a)v_E \end{array} \right\rangle = \\
&= \left\langle \begin{array}{l} 1 + y c_{2\gamma} D_S D_O \\ s_{2\varepsilon} D_O s_{2\varepsilon-2\gamma} + y D_S [s_{2\varepsilon}^2 - s_{2\varepsilon} s_{2\gamma} W_O c_{2\varepsilon-2\gamma} - x c_{2\varepsilon} s_{2\gamma} Z_O S_O] \\ -c_{2\varepsilon} D_O s_{2\varepsilon-2\gamma} - y D_S [s_{2\varepsilon} c_{2\varepsilon} - c_{2\varepsilon} s_{2\gamma} W_O c_{2\varepsilon-2\gamma} + x s_{2\varepsilon} s_{2\gamma} Z_O S_O] \\ -x \{ D_O c_{2\varepsilon-2\gamma} + y D_S [c_{2\varepsilon} + s_{2\gamma} W_O s_{2\varepsilon-2\gamma}] \} \end{array} \left| \begin{array}{l} i_E \\ a q_E \\ -a u_E \\ (1-2a)v_E \end{array} \right. \right\rangle
\end{aligned} \tag{160}$$

• **Special cases:** Without receiver optics rotation γ we get from Eq. (E.19) and Eq.(160) the calibration signals Eq. (161) and the gain ratios Eq. (162).

$$\gamma = 0 \Rightarrow$$

$$\begin{aligned}
\frac{I_S}{\eta_S T_S T_O T_{QW} F_{11} T_E I_L} &= \frac{\langle \mathbf{A}_S(y, 0) | \mathbf{M}_{QW}(x45^\circ + \varepsilon) | \mathbf{F}(a) \mathbf{M}_E I_L \rangle}{T_S T_O T_{QW} F_{11} T_E I_L} = \\
&= \left\langle \begin{array}{l} 1 + y D_S D_O \\ s_{2\varepsilon}^2 (D_O + y D_S) \\ -c_{2\varepsilon} s_{2\varepsilon} (D_O + y D_S) \\ -x c_{2\varepsilon} (D_O + y D_S) \end{array} \left| \begin{array}{l} i_E \\ a q_E \\ -a u_E \\ (1-2a)v_E \end{array} \right. \right\rangle = \\
&= (1 + y D_S D_O) i_E + (D_O + y D_S) [s_{2\varepsilon} a (s_{2\varepsilon} q_E + c_{2\varepsilon} u_E) - x c_{2\varepsilon} (1-2a) v_E]
\end{aligned} \tag{161}$$

$$\gamma = 0 \Rightarrow$$

$$\frac{\eta^*}{\eta} = \frac{(1 + y D_R D_O) i_E + (D_O + y D_R) [s_{2\varepsilon} a (s_{2\varepsilon} q_E + c_{2\varepsilon} u_E) - x c_{2\varepsilon} (1-2a) v_E]}{(1 + y D_T D_O) i_E + (D_O + y D_T) [s_{2\varepsilon} a (s_{2\varepsilon} q_E + c_{2\varepsilon} u_E) - x c_{2\varepsilon} (1-2a) v_E]} \tag{162}$$

• Eq.(162) with a cleaned PBS (Suppl. S.10.10) becomes Eq. (163), and without calibrator rotation ε Eq.(162) becomes Eq. (164).

$$\gamma = 0, D_T = +1, D_R = -1 \Rightarrow$$

$$\frac{\eta^*}{\eta} = \frac{1 - y D_O i_E - y [s_{2\varepsilon} a (s_{2\varepsilon} q_E + c_{2\varepsilon} u_E) - x c_{2\varepsilon} (1-2a) v_E]}{1 + y D_O i_E + y [s_{2\varepsilon} a (s_{2\varepsilon} q_E + c_{2\varepsilon} u_E) - x c_{2\varepsilon} (1-2a) v_E]} \tag{163}$$

$$\gamma = \varepsilon = 0 \Rightarrow$$

$$\frac{\eta^*}{\eta} = \frac{(1 + y D_R D_O) i_E - x (D_O + y D_R) (1-2a) v_E}{(1 + y D_T D_O) i_E - x (D_O + y D_T) (1-2a) v_E} \tag{164}$$

• With a cleaned analyser and without calibrator rotation ε the gain ratios in Eq.(162) become Eq. (165) and for the $\Delta 90$ -calibration Eq. (166).

$$\gamma = \varepsilon = 0, D_T = +1, D_R = -1 \Rightarrow$$

$$\frac{\eta^*}{\eta} = \frac{1 - yD_O i_E + xy(1-2a)v_E}{1 + yD_O i_E - xy(1-2a)v_E} \quad (165)$$

$$\frac{\eta_{\Delta 90}^*}{\eta} = \frac{1 - yD_O}{1 + yD_O} \quad (166)$$

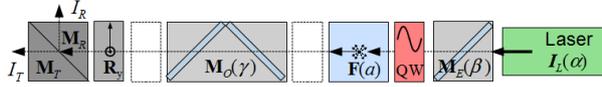
• Eq. (165) can be rearranged with Eq. (166) to Eq. (167), from which we get the degree of circular polarisation v_E/i_E of the beam behind the emitter optics in Eq. (168). The atmospheric polarisation parameter a must be estimated from a standard measurement, and if we use an atmospheric range without aerosols it becomes $a \approx 1$. While v_{in} in Eq. (158) includes the mostly unknown retardation terms of the receiving optics, v_E in Eq. (168) is free of them and hence a better estimation for the elliptical polarisation of the laser.

$$\gamma = \varepsilon = 0, D_T = +1, D_R = -1 \Rightarrow$$

$$\frac{\eta^*}{\eta_{\Delta 90}^*} = \frac{i_E + xy(1-2a)v_E}{i_E - xy(1-2a)v_E} \Rightarrow \quad (167)$$

$$\frac{v_E}{i_E} = \frac{1}{xy(1-2a)} \frac{\eta^* - \eta_{\Delta 90}^*}{\eta^* + \eta_{\Delta 90}^*} \quad (168)$$

9.3 Calibration with an ideal $\lambda/4$ plate behind the emitter optics



$$\begin{aligned} I_S &= \eta_S \mathbf{M}_S \mathbf{R}(y) \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_{QW}(x45^\circ + \varepsilon) \mathbf{M}_E(\beta) I_L(\alpha) = \\ &= \eta_S \mathbf{A}_S(y, \gamma, a) \mathbf{M}_{QW}(x45^\circ + \varepsilon) I_{in}(\beta, \alpha) \end{aligned} \quad (169)$$

With Eq. (D.13) for the analyser part \mathbf{A}_S , an ideal $\lambda/4$ plate \mathbf{M}_{QW} Eq. (S.10.17.3), and with an input Stokes vector I_{in} from Eq.(E.8) we get the general calibration signals I_S in Eq. (170).

$$\frac{I_S}{\eta_S T_S T_O F_{11} T_{QW} T_E I_L} = \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{F}(a) | \mathbf{M}_{QW}(x45^\circ + \varepsilon) | \mathbf{M}_E I_L \rangle}{T_S T_O F_{11} T_{QW} T_E I_L} =$$

$$= \left\langle \begin{array}{c} 1 + yc_{2\gamma} D_S D_O c \\ a [c_{2\gamma} D_O + yD_S (1 - s_{2\gamma}^2 W_O)] \\ -as_{2\gamma} (D_O + yc_{2\gamma} D_S W_O) \\ -(1-2a) ys_{2\gamma} D_S Z_O S_O \end{array} \right| \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & s_{2\varepsilon}^2 & -s_{2\varepsilon} c_{2\varepsilon} & -xc_{2\varepsilon} \\ 0 & -s_{2\varepsilon} c_{2\varepsilon} & c_{2\varepsilon}^2 & -xs_{2\varepsilon} \\ 0 & xc_{2\varepsilon} & xs_{2\varepsilon} & 0 \end{array} \right) \left| \begin{array}{c} i_E \\ q_E \\ u_E \\ v_E \end{array} \right\rangle \quad (170)$$

• **Special cases:** Equivalent to Sect.9.2 we get from Eq. (170) without receiver optics rotation γ the calibration signals in Eq. (171).

with $\gamma = 0 \Rightarrow$

$$\frac{I_S}{\eta_S T_S T_O T_{QW} F_{11} T_E I_L} = \frac{\langle \mathbf{A}_S(y, 0, a) | \mathbf{M}_{QW}(x45^\circ + \varepsilon) | \mathbf{M}_E \mathbf{I}_L \rangle}{T_S T_O T_{QW} F_{11} T_E I_L} = \left\langle \begin{array}{c} 1 + yD_S D_O \\ s_{2\varepsilon}^2 a (D_O + yD_S) \\ -c_{2\varepsilon} s_{2\varepsilon} a (D_O + yD_S) \\ -x c_{2\varepsilon} a (D_O + yD_S) \end{array} \middle| \begin{array}{c} i_E \\ q_E \\ u_E \\ v_E \end{array} \right\rangle \quad (171)$$

• From Eq. (171) without calibrator rotation ε we get the gain ratios Eq. (172), • with additionally a cleaned analyser we get Eq. (173), and with the corresponding $\Delta 90$ -calibration Eq. (174).

with $\gamma = \varepsilon = 0 \Rightarrow$

$$\frac{\eta^*}{\eta} = \frac{(1 + yD_R D_O) i_E - xa(D_O + yD_R) v_E}{(1 + yD_T D_O) i_E - xa(D_O + yD_T) v_E} \quad (172)$$

with $\gamma = \varepsilon = 0, D_T = +1, D_R = -1 \Rightarrow$

$$\frac{\eta^*}{\eta} = \frac{1 - yD_O i_E + xyav_E}{1 + yD_O i_E - xyav_E} \quad (173)$$

$$\frac{\eta_{\Delta 90}^*}{\eta} = \frac{1 - yD_O}{1 + yD_O} \quad (174)$$

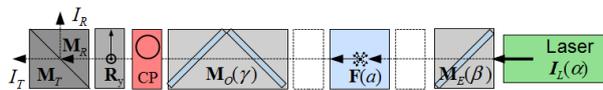
• Eq. (173) can be rearranged with Eq. (174) to Eq. (175) from which we get the degree of circular polarisation v_E/i_E of the beam behind the emitter optics if the atmospheric polarisation parameter a is known, as e.g. when we use the lidar signals from an atmospheric range without aerosols where $a \approx 1$.

$$\frac{\eta^*}{\eta_{\Delta 90}^*} = \frac{i_E + xyav_E}{i_E - xyav_E} \Rightarrow \frac{v_E}{i_E} = \frac{1}{xya} \frac{\eta^* - \eta_{\Delta 90}^*}{\eta^* + \eta_{\Delta 90}^*} \quad (175)$$

10 Calibration with a circular polariser (CP)

The use of a circular polariser seems to be ideal for the calibration, but the uncertainties of a real circular polariser are usually not provided by manufacturers and might be difficult to determine. A real CP is mostly a combination of a linear polariser followed by a QWP at $z45^\circ$ ($z = \pm 1$) (see Suppl. S.10.18), and therefore it combines the uncertainties of both (see Sects. 8 and 9). Before the results of a circularly polarising calibrator can be trusted, the diattenuation of the linear polariser and the phase shift uncertainties should be determined and the error assessment performed using the general Eq. (C.10) for the calibration signals. If we consider all possible error terms, the Müller matrix for a real CP becomes too complex for this investigation, wherefore we assume a circular polariser with phase shift error ω but with an ideal linear polariser from Eq. (S.10.18.4) in the following in order to show the possibilities of this calibrator.

10.1 Calibration with a circular polariser before the polarising beam-splitter



$$\begin{aligned}
\mathbf{I}_S &= \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_{CP}(z, x45^\circ + \varepsilon, \omega) \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha) = \\
&= \eta_S \mathbf{A}_S(y) \mathbf{M}_{CP}(z, x45^\circ + \varepsilon, \omega) \mathbf{I}_{in}(\gamma, a, \beta, \alpha)
\end{aligned} \tag{176}$$

With \mathbf{A}_S from Eq. (D.5), the circularly polarising calibrator \mathbf{M}_{CP} with retardation error ω from Eq. (S.10.18.4), and the input Stokes vectors \mathbf{I}_{in} from App. E.4 we get Eq. (177) for the calibration signals I_S . As the last term of Eq. (177) is independent of D_S , it cancels out in the gain ratios Eq. (178), which is therefore independent of the input Stokes vector, but still includes ε and ω terms.

$$\begin{aligned}
\frac{I_S}{\eta_S T_S T_{CP} I_{in}} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{M}_{CP}(z, x45^\circ + \varepsilon, \omega) | \mathbf{I}_{in} \rangle}{T_S T_{CP} I_{in}} = \\
&= \left\langle \begin{array}{c|c} 1 & 1 \\ yD_S & xS_{2\varepsilon}S_\omega \\ 0 & -x\mathbf{c}_{2\varepsilon}S_\omega \\ 0 & z\mathbf{c}_\omega \end{array} \right\rangle \left\langle \begin{array}{c|c} 1 & i_{in} \\ -xS_{2\varepsilon} & q_{in} \\ x\mathbf{c}_{2\varepsilon} & u_{in} \\ 0 & v_{in} \end{array} \right\rangle = [1 + xyD_S S_{2\varepsilon} S_\omega] \left\langle \begin{array}{c|c} 1 & i_{in} \\ -xS_{2\varepsilon} & q_{in} \\ x\mathbf{c}_{2\varepsilon} & u_{in} \\ 0 & v_{in} \end{array} \right\rangle = \\
&= (1 + xyD_S S_{2\varepsilon} S_\omega) [i_{in} - x(S_{2\varepsilon} q_{in} - \mathbf{c}_{2\varepsilon} u_{in})]
\end{aligned} \tag{177}$$

$$\frac{\eta^*}{\eta} = \frac{1 + xyD_R S_{2\varepsilon} S_\omega}{1 + xyD_T S_{2\varepsilon} S_\omega} \tag{178}$$

• If ω is zero, we have an ideal circular polariser with which we get the gain ratio independently of ε , and if ε is zero ω doesn't matter (Eq. (179)).

$$\omega = 0 \vee \varepsilon = 0 \Rightarrow$$

$$\frac{\eta^*}{\eta} = 1 \tag{179}$$

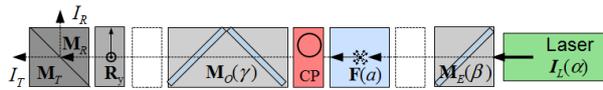
• With a cleaned analyser we get from Eq. (178) Eqs. (180) and (181), which show that the deviations of the gain ratios are fully compensated by the $\Delta 90$ -calibration. ω can be determined by means of the successive approximation in Sect. 11, Eqs. (198) ff.

$$D_T = +1, D_R = -1 \Rightarrow$$

$$\frac{\eta^*}{\eta} = \frac{1 - xys_{2\varepsilon}S_\omega}{1 + xys_{2\varepsilon}S_\omega} \tag{180}$$

$$\frac{\eta_{\Delta 90}^*}{\eta} = 1 \tag{181}$$

10.2 Calibration with a circular polariser before the receiving optics



$$\begin{aligned}
\mathbf{I}_S &= \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{M}_{CP}(z, x45^\circ + \varepsilon, \omega) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha) = \\
&= \eta_S \mathbf{A}_S(y, \gamma) \mathbf{M}_{CP}(z, x45^\circ + \varepsilon, \omega) \mathbf{I}_{in}(a, \beta, \alpha)
\end{aligned} \tag{182}$$

With \mathbf{A}_S from App. D.2, \mathbf{M}_{CP} with retardation error ω from Eq. (S.10.18.4), and \mathbf{I}_m from App. E.3 we get Eq. (183) for the calibration signals I_S .

$$\begin{aligned} \frac{I_S}{\eta_S T_S T_{CP} T_O F_{11} T_E I_L} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) | \mathbf{M}_{CP}(z, x45^\circ + \varepsilon, \omega) | \mathbf{F}(a) \mathbf{M}_E \mathbf{I}_L \rangle}{T_S T_O T_{CP} F_{11} T_E I_L} = \\ &= \left\langle \begin{array}{c} 1 + y c_{2\gamma} D_S D_O \\ c_{2\gamma} D_O + y D_S (1 - s_{2\gamma}^2 W_O) \\ s_{2\gamma} (D_O + y c_{2\gamma} D_S W_O) \\ -y s_{2\gamma} D_S Z_O S_O \end{array} \left\| \begin{array}{c} 1 \\ x s_{2\varepsilon} s_\omega \\ -x c_{2\varepsilon} s_\omega \\ z c_\omega \end{array} \right. \left. \begin{array}{c} 1 \\ -x s_{2\varepsilon} \\ x c_{2\varepsilon} \\ 0 \end{array} \right\| \begin{array}{c} i_E \\ a q_E \\ -a u_E \\ (1 - 2a) v_E \end{array} \right\rangle = \\ &= \left\{ \begin{array}{l} 1 + y D_S (c_{2\gamma} D_O - s_{2\gamma} Z_O S_O z c_\omega) + \\ + x s_\omega [D_O s_{2\varepsilon - 2\gamma} + y D_S (s_{2\varepsilon} - s_{2\gamma} W_O c_{2\varepsilon - 2\gamma})] \end{array} \right\} \left[i_E - x a (s_{2\varepsilon} q_E + c_{2\varepsilon} u_E) \right] \end{aligned} \quad (183)$$

As the last term of Eq. (183) is independent of D_S , it cancels out in the gain ratio. However, as long as the receiver optics rotation γ doesn't vanish, the gain ratios include deviations which don't cancel with the $\Delta 90$ -calibration, even if we • use a cleaned analyser (Eq. (184)) and • additionally an ideal circular polariser (Eq. (185)) or • without calibrator error ε (Eq. (186)).

$$D_T = +1, D_R = -1 \Rightarrow$$

$$\frac{\eta^*}{\eta} = \frac{1 - y (c_{2\gamma} D_O - s_{2\gamma} Z_O S_O z c_\omega) + x s_\omega [D_O s_{2\varepsilon - 2\gamma} - y (s_{2\varepsilon} - s_{2\gamma} W_O c_{2\varepsilon - 2\gamma})]}{1 + y (c_{2\gamma} D_O - s_{2\gamma} Z_O S_O z c_\omega) + x s_\omega [D_O s_{2\varepsilon - 2\gamma} + y (s_{2\varepsilon} - s_{2\gamma} W_O c_{2\varepsilon - 2\gamma})]} \quad (184)$$

$$D_T = +1, D_R = -1, \omega = 0 \Rightarrow$$

$$\frac{\eta^*}{\eta} = \frac{1 - y (c_{2\gamma} D_O - s_{2\gamma} Z_O S_O)}{1 + y (c_{2\gamma} D_O - s_{2\gamma} Z_O S_O)} \quad (185)$$

$$D_T = +1, D_R = -1, \varepsilon = 0 \Rightarrow$$

$$\frac{\eta^*}{\eta} = \frac{1 - y (c_{2\gamma} D_O - s_{2\gamma} Z_O S_O z c_\omega) - x s_\omega s_{2\gamma} [D_O - y W_O c_{2\gamma}]}{1 + y (c_{2\gamma} D_O - s_{2\gamma} Z_O S_O z c_\omega) - x s_\omega s_{2\gamma} [D_O + y W_O c_{2\gamma}]} \quad (186)$$

• From Eq. (183) without receiver optics rotation γ we get Eq. (187), and

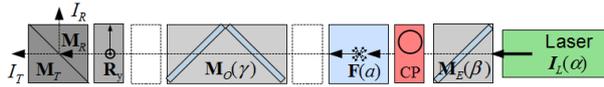
• with additionally a cleaned analyser the Eqs. (188) and (189) are the same as in the previous sections but with the pre-factor of Eq. (189).

$$\gamma = 0 \Rightarrow \frac{\eta^*}{\eta} = \frac{1 + y D_R D_O + x s_\omega s_{2\varepsilon} (D_O + y D_R)}{1 + y D_T D_O + x s_\omega s_{2\varepsilon} (D_O + y D_T)} \quad (187)$$

$$\gamma = 0, D_T = +1, D_R = -1 \Rightarrow \frac{\eta^*}{\eta} = \frac{1 - y D_O}{1 + y D_O} \frac{1 - x y s_\omega s_{2\varepsilon}}{1 + x y s_\omega s_{2\varepsilon}} \quad (188)$$

$$[\gamma = 0, D_T = +1, D_R = -1] \wedge [\omega = 0 \vee \varepsilon = 0] \Rightarrow \frac{\eta^*}{\eta} = \frac{1 - y D_O}{1 + y D_O} \quad (189)$$

10.3 Calibration with a circular polariser behind the emitter optics



$$\begin{aligned}
 I_S &= \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_{CP}(z, x45^\circ + \varepsilon) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha) = \\
 &= \eta_S \mathbf{A}_S(y, \gamma, a) \mathbf{M}_{CP}(z, x45^\circ + \varepsilon) \mathbf{I}_E(\beta, \alpha)
 \end{aligned} \tag{190}$$

With \mathbf{A}_S from App. D.3, \mathbf{M}_{CP} with retardation error ω from Eq. (S.10.18.4), and \mathbf{I}_{in} from App. E.2 we get Eq. (191) for the calibration signals I_S , which differs from Eq. (183) in the last section just by the prefactors depending on the atmospheric polarisation parameter a . The same holds for the gain ratios derived with a • cleaned analyser in Eqs. (192) compared to Eq. (184) and all the subsequent derivations there.

$$\begin{aligned}
 \frac{I_S}{\eta_S T_S T_O F_{11} T_{CP} T_E I_L} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{F}(a) | \mathbf{M}_{CP}(z, x45^\circ + \varepsilon, \omega) | \mathbf{M}_E \mathbf{I}_L \rangle}{T_S T_O F_{11} T_{CP} T_E I_L} = \\
 &= \left\langle \begin{array}{c} 1 + y c_{2\gamma} D_S D_O c \\ a [c_{2\gamma} D_O + y D_S (1 - s_{2\gamma}^2 W_O)] \\ -a s_{2\gamma} (D_O + y c_{2\gamma} D_S W_O) \\ -(1 - 2a) y s_{2\gamma} D_S Z_O S_O \end{array} \right| \left| \begin{array}{c} 1 \\ x s_{2\varepsilon} s_\omega \\ -x c_{2\varepsilon} s_\omega \\ z c_\omega \end{array} \right\rangle \left\langle \begin{array}{c} 1 \\ -x s_{2\varepsilon} \\ x c_{2\varepsilon} \\ 0 \end{array} \right| \left| \begin{array}{c} i_E \\ q_E \\ u_E \\ v_E \end{array} \right\rangle = \\
 &= \left\{ \begin{array}{l} 1 + y D_S (c_{2\gamma} D_O - (1 - 2a) s_{2\gamma} Z_O S_O z c_\omega) + \\ + x a s_\omega [D_O s_{2\varepsilon - 2\gamma} + y D_S (s_{2\varepsilon} - s_{2\gamma} W_O c_{2\varepsilon - 2\gamma})] \end{array} \right\} \left[i_E - x (s_{2\varepsilon} q_E + c_{2\varepsilon} u_E) \right]
 \end{aligned} \tag{191}$$

$$D_T = +1, D_R = -1 \Rightarrow$$

$$\frac{\eta^*}{\eta} = \frac{1 - y (c_{2\gamma} D_O - s_{2\gamma} (1 - 2a) Z_O S_O z c_\omega) + x a s_\omega [D_O s_{2\varepsilon - 2\gamma} - y (s_{2\varepsilon} - s_{2\gamma} W_O c_{2\varepsilon - 2\gamma})]}{1 + y (c_{2\gamma} D_O - s_{2\gamma} (1 - 2a) Z_O S_O z c_\omega) + x a s_\omega [D_O s_{2\varepsilon - 2\gamma} + y (s_{2\varepsilon} - s_{2\gamma} W_O c_{2\varepsilon - 2\gamma})]} \tag{192}$$

11 Determination of the calibrator rotation ε

The calibration measurements can be used to determine and consequentially correct the calibrator rotation ε , which is especially important for the rotation calibrator (Sect. 7), because here the rotation error ε is also present in the standard measurements and has to be corrected, either mechanically before the measurements or analytically after the measurements. If the $\pm 45^\circ$ calibration measurements can be described or approximated by Eq. (193) with $f(y, \dots)$ being a function of any parameter but not of x and ε , it is possible to estimate the calibrator rotation ε by means of the relative difference of the $\pm 45^\circ$ gain ratios as in Eq. (194) and using the tangent half-angle substitution (Suppl. S.12.1) to achieve ε from Eq. (195). Note: η is assumed to be unknown.

$$\frac{\eta^*}{\eta} = f(y, \dots) \frac{1 + x s_{2\varepsilon}}{1 - x s_{2\varepsilon}} \tag{193}$$

$$Y(\varepsilon) \equiv \frac{\eta^*(y, +45^\circ + \varepsilon) - \eta^*(y, -45^\circ + \varepsilon)}{\eta^*(y, +45^\circ + \varepsilon) + \eta^*(y, -45^\circ + \varepsilon)} = \frac{\frac{1+s_{2\varepsilon}}{1-s_{2\varepsilon}} - \frac{1-s_{2\varepsilon}}{1+s_{2\varepsilon}}}{\frac{1+s_{2\varepsilon}}{1-s_{2\varepsilon}} + \frac{1-s_{2\varepsilon}}{1+s_{2\varepsilon}}} = \frac{2s_{2\varepsilon}}{1+s_{2\varepsilon}^2} \quad (194)$$

$$\varepsilon(Y) = 0.5 * \arcsin \left[\tan \left(0.5 * \arcsin [Y] \right) \right] \quad (195)$$

With the assumption $\sin(2\varepsilon) \ll 1$ we get a good approximation for ε in the simple Eq. (196), which deviates by about 5% at $\varepsilon \approx 6^\circ$ and $Y(\varepsilon) \approx 0.4$.

$$s_{2\varepsilon} \ll 1 \Rightarrow Y(\varepsilon) \approx 2s_{2\varepsilon} \Rightarrow \varepsilon \approx 0.25 * Y \quad (196)$$

Eq. (193) is applicable in Eqs. (130) and (137) for the linear polariser calibrator, and it is a good approximation for Eq (144) if the atmospheric polarisation parameter $a \approx 1$. For the rotation calibration before the receiving optics (Sect. 7.2, Eq.(117)) we have to assume that $a \approx 1$ and additionally that the laser beam behind the emitter optics is horizontal-linearly polarised. Eq.(117) can then be approximated by Eq. (197).

$$\text{with } \gamma = 0, D_T = +1, D_R = -1, i_E = q_E = 1, u_E = v_E = 0, a \approx 1 \Rightarrow \frac{\eta^*}{\eta} \approx \frac{1 - yD_O}{1 + yD_O} \frac{1 + xs_{2\varepsilon}}{1 - xs_{2\varepsilon}} \quad (197)$$

If instead of Eq.(193) we have a form as Eq. (198) (see Suppl. S.12.1), we get Eqs. (199) and (200). If ε is known, Eq.(200) can be solved for K , which yields Eq. (201).

$$\frac{\eta^*}{\eta} = f(y, \dots) \frac{1 + Kxs_{2\varepsilon}}{1 - Kxs_{2\varepsilon}} \quad \text{with } K \leq 1 \quad (198)$$

$$Y(\varepsilon, K) \equiv \frac{\eta^*(y, +45^\circ + \varepsilon, K) - \eta^*(y, -45^\circ + \varepsilon, K)}{\eta^*(y, +45^\circ + \varepsilon, K) + \eta^*(y, -45^\circ + \varepsilon, K)} = \frac{2Ks_{2\varepsilon}}{1 + K^2s_{2\varepsilon}^2} \quad (199)$$

$$\varepsilon = \frac{1}{2} \arcsin \left[\frac{1}{K} \tan \left(\frac{\arcsin [Y(\varepsilon, K)]}{2} \right) \right] \quad (200)$$

$$K = \left[\frac{1}{\sin 2\varepsilon} \tan \left(\frac{\arcsin [Y(\varepsilon, K)]}{2} \right) \right] \quad (201)$$

If the true ε and K are unknown, we can retrieve them by successive approximation. With $K < 1$ we find as a first approximation ε_1 from Eq. (202) and make the next calibration measurement after adjusting the calibrator rotation by $-\varepsilon_1$, which results in the actual position $(\varepsilon - \varepsilon_1)$ and the corresponding Eq. (203).

$$\varepsilon_1 = \frac{1}{2} \arcsin \left[\tan \left(\frac{\arcsin [Y(\varepsilon, K)]}{2} \right) \right] < \varepsilon \quad (202)$$

$$Y(\varepsilon - \varepsilon_1, K) = \frac{2Ks_{2(\varepsilon - \varepsilon_1)}}{1 + K^2s_{2(\varepsilon - \varepsilon_1)}^2} \quad (203)$$

Using the calibration measurements at the two positions ε and $(\varepsilon - \varepsilon_1)$ with Eqs.(199) and (203), we get an estimation of the true ε with Eq. (205) derived from the ratio in Eq. (204).

$$\frac{Y(\varepsilon - \varepsilon_1, K)}{Y(\varepsilon, K)} = \frac{(1 + K^2 s_{2\varepsilon}^2) 2K s_{2(\varepsilon - \varepsilon_1)}}{(1 + K^2 s_{2(\varepsilon - \varepsilon_1)}^2) 2K s_{2\varepsilon}} \approx \frac{s_{2(\varepsilon - \varepsilon_1)}}{s_{2\varepsilon}} \approx \frac{(\varepsilon - \varepsilon_1)}{\varepsilon} = 1 - \frac{\varepsilon_1}{\varepsilon} \quad (204)$$

$$\varepsilon \approx \frac{Y(\varepsilon, K)}{Y(\varepsilon, K) - Y(\varepsilon - \varepsilon_1, K)} \varepsilon_1 \quad (205)$$

Finally, with known ε , we can use Eq. (201) to estimate K .

12 Determination of the rotation α of the plane of polarisation of the emitted laser beam.

The orientation of the plane of polarisation of the laser beam is in general specified by manufacturers just as *vertical* or *horizontal*, without specifying the reference and the accuracy. Furthermore, the assembly of the laser with the telescope and the receiver optics in a lidar system can often not be done with similar accuracy as the assembly of the optical elements in the receiver optics, and the necessary alignment mechanisms for the tilt between the laser and telescope axes additionally introduces variability and uncertainty. On top of that, the adjustments may change after every laser maintenance. Therefore it is desirable to determine the laser rotation once in a while.

Using the calibrator equations for the calibrator before the receiver optics from App. C with an analyser without receiver optics rotation ($\gamma = 0$; Eq. (D.8)), i.e.

$$\begin{aligned} \gamma = 0^\circ &\Rightarrow \langle \mathbf{A}_S | (y, 0^\circ) \rangle = \langle \mathbf{M}_S \mathbf{R}_y | \mathbf{M}_O(0^\circ) = T_O T_S \langle 1 + y D_S D_O \quad D_O + y D_S \quad 0 \quad 0 | = \\ &= \langle \mathbf{M}_{S y O}(0^\circ) | = T_{S y O} \langle 1 \quad D_{S y O} \quad 0 \quad 0 | \\ &\Rightarrow A_S^3 = A_S^4 = 0 \end{aligned}$$

with elliptically polarised emitted laser light as Eq. (E.25),

$$\frac{\mathbf{I}_{in}(a, b, \alpha)}{I_{in}} = \frac{\mathbf{F}(a) \mathbf{I}_E}{F_{11} T_E I_L} = \left| 1 \quad abc_{2\alpha} \quad -abs_{2\alpha} \quad (1 - 2a) \sqrt{1 - b^2} \right\rangle,$$

and with ideal calibrators, we get the signals for the four ideal calibrator types in Eqs. (206) to (209).

$$\begin{aligned} \frac{I_S}{\eta_S I_{in}} &= \frac{\langle \mathbf{A}_S | \mathbf{M}_{rot}(x45^\circ + \varepsilon, h) | \mathbf{I}_{in} \rangle}{I_{in}} = \\ &= A_S^1 j_{in} + A_S^4 h v_{in} - x \left[(s_{2\varepsilon} A_S^2 - c_{2\varepsilon} A_S^3) q_{in} + (c_{2\varepsilon} A_S^2 + s_{2\varepsilon} A_S^3) h u_{in} \right] = \\ &= T_{S y O} (1 - x a b D_{S y O} s_{2\varepsilon - h 2\alpha}) \end{aligned} \quad (206)$$

$$D_p = 1 \Rightarrow$$

$$\begin{aligned} \frac{I_S}{\eta_S T_P I_{in}} &= \frac{\langle \mathbf{A}_S | \mathbf{M}_P(x45^\circ + \varepsilon) | \mathbf{I}_{in} \rangle}{T_P I_{in}} = \left[A_S^1 + x (c_{2\varepsilon} A_S^3 - s_{2\varepsilon} A_S^2) \right] \left[i_{in} + x (c_{2\varepsilon} u_{in} - s_{2\varepsilon} q_{in}) \right] = \\ &= T_{S y O} (1 - x s_{2\varepsilon} D_{S y O}) (1 - x a b s_{2\alpha + 2\varepsilon}) = T_{S y O} \left[1 + a b D_{S y O} s_{2\varepsilon} s_{2\alpha + 2\varepsilon} - x (s_{2\varepsilon} D_{S y O} + a b s_{2\alpha + 2\varepsilon}) \right] \end{aligned} \quad (207)$$

$$\omega = 0 \Rightarrow$$

$$\begin{aligned} \frac{I_S}{\eta_S T_{QW} I_{in}} &= \frac{\langle \mathbf{A}_S | \mathbf{M}_{QW}(x45^\circ + \varepsilon, 0) | \mathbf{I}_{in} \rangle}{T_{QW} I_{in}} = \\ &= A_S^1 i_{in} - (s_{2\varepsilon} A_S^2 - c_{2\varepsilon} A_S^3) (s_{2\varepsilon} q_{in} - c_{2\varepsilon} u_{in}) - x \left[A_S^4 (c_{2\varepsilon} q_{in} + s_{2\varepsilon} u_{in}) + (c_{2\varepsilon} A_S^2 + s_{2\varepsilon} A_S^3) v_{in} \right] = \\ &= T_{SyO} \left[1 - ab D_{SyO} s_{2\varepsilon} s_{2\varepsilon+2\alpha} + x D_{SyO} c_{2\varepsilon} (1 - 2a) \sqrt{1 - b^2} \right] \end{aligned} \quad (208)$$

$$\omega = 0, D_P = 1 \Rightarrow$$

$$\begin{aligned} \frac{I_S}{\eta_S T_{CP} I_{in}} &= \frac{\langle \mathbf{A}_S | \mathbf{M}_{CP}(z, x45^\circ + \varepsilon) | \mathbf{I}_{in} \rangle}{T_{CP} I_{in}} = (A_S^1 + z A_S^4) (i_{in} - x (c_{2\varepsilon} u_{in} - s_{2\varepsilon} q_{in})) = \\ &= T_{SyO} (1 + x a b s_{2\alpha+2\varepsilon}) \end{aligned} \quad (209)$$

Eqs. (206) and (209) are of the type of Eq. (193), wherefore the solutions described in Sect. 11 can be applied, but only to determine $\varepsilon \pm \alpha$. In order to determine α alone, ε must be known, or a series of measurements with variable ε are fitted to the gain ratios η^* formulated with one of the Eqs. (206) to (209), as explained by Alvarez et al. (2006).

Furthermore, for the case of the linear polariser calibrator (Eq. (207)), an unpolarised light source (i.e. $i_{in} = 1$ and $q_{in} = u_{in} = v_{in} = 0$) before the receiver optics / telescope gives Eq. (210) from Eq. (207), which is of the type of Eq. (193), and with a cleaned analyser $D_{SyO} = \pm 1$.

$$D_P = 1, i_{in} = 1, q_{in} = u_{in} = v_{in} = 0 \Rightarrow$$

$$\frac{I_S}{\eta_S T_P I_{in}} = \frac{\langle \mathbf{A}_S | \mathbf{M}_P(x45^\circ + \varepsilon) | \mathbf{I}_{in} \rangle}{T_P I_{in}} = T_{SyO} (1 - x s_{2\varepsilon} D_{SyO}) \quad (210)$$

13 Assumptions and constraints of the model

1. The correction of the standard signals (Sect. 4) and of the calibration factor (Sects. 5 ff) are only applicable in scattering ranges without aerosol or with randomly oriented, non-spherical particles with rotation and reflection symmetry as described in Sect. 2.1, and not for clouds with oriented particles as in cirrus and rain clouds (Kaul et al. 2004; Hayman et al. 2014; Volkov et al. 2015). However, the scattering volume for the calibration measurements can be chosen to avoid oriented particles in the calibration range, and then the calibration corrections in Sects. 5 to 10 can be applied for the retrieval of the calibration factor $\eta = (\eta_R T_R) / (\eta_T T_T)$, which itself is general for the considered types of lidar set-ups in Fig. (1).

2. We assume that the extinction in the range between the lidar and the scattering volume is polarisation independent and that signal contributions due to multiple scattering can be neglected.

3. We assume that the atmospheric depolarisation in the calibration range does not change between the two measurements of the $\Delta 90$ -calibrations. This can be verified by comparison of standard measurements before and after and possibly between the two calibration measurements.

4. Not considered are range dependent effects as the overlap function and the range dependent transmission and polarisation of interference filters and dichroic beam-splitters, which is caused by the range dependent incident angles on the optics.

5. We assume that the optical elements of the lidar do not depolarize. Such depolarization can be caused by optical elements inside the emitter and receiver optics, which are not well

aligned with their optical axes, and by variable retardation or diattenuation over the aperture of optical elements, for example due to crystalline (e.g. CaF₂ and MgF₂ lenses) or stress birefringence. The latter can be present in all optical elements if they are inappropriately restrained in their holders. Larger optics, as e.g. telescope windows, can exhibit inherent stress birefringence due to annealing and/or their own weight. Such optical elements can easily be visually inspected by means of crossed polarising sheet filters before and after the sample. Furthermore, non-parallel (converging or diverging) incident beams on optics with polarisation effects depending on the incidence angle will cause depolarisation. The manufacturers specification of dedicated polarisation optics should be sufficient to determine the maximum allowable divergence of the incident beam, but, for example, the coatings of 90° reflecting mirrors in Newtonian telescopes are usually not sufficiently specified to determine their polarisation effects. The depolarising effects of optics can additionally depend on the state of polarisation of the incident beam.

14 Summary and conclusions

The presented equations can be used to analyse the effects of polarising optics of a variety of lidar systems and to assess the accuracy and error of several calibration techniques. From the parameters of the optics, which are either given by the manufacturers or measured (some can be measured by means of procedures described in this work), we determine the general signal parameters G_S and H_S in Sect. 4 for the correction of the measured signals and the derived linear depolarisation ratio regarding polarisation effects of the lidar system. They are independent of the atmospheric depolarisation. Additionally we need calibration measurements for the unpolarised gain ratio of the cross to the parallel signal. They can depend on the atmospheric depolarisation and must also be corrected for polarisation effects of the lidar system as described in Sects. 5 to 10. We show how this dependence can be avoided or minimized and presented correction formulas for the gain ratio for all cases.

Major findings are, that a cleaned analyser and no rotation of the receiving optics with respect to the laser polarisation avoid many error terms and allow to determine and correct other misalignments and the optics diattenuation, and that the $\Delta 90^\circ$ -calibration can decrease the error of a single $\pm 45^\circ$ calibration into insignificance.

We show that a linear polariser as calibrator should have a very good extinction ratio in order to avoid large calibration errors (Eq. (126)). The advantage of a sheet polariser (and $\lambda/4$ sheet filters) is its tenuity, wherefore it can be included in many existing lidar systems with minimal space requirement, for example with a sheet holder as shown in Fig. 5. Such a sheet holder guarantees an accurate $\Delta 90^\circ$ rotation of the sheet, wherefore the absolute accuracy of the 45° orientation is not important. Together with an existing calibration technique or inserted at different positions, the filter holder can be used to determine the diattenuation of the optics between the two positions (see Eqs. (131) and (138) / (145)). Furthermore, the determination of the calibration factor with an ideal linear polariser calibrator is always independent of changes of the input light and hence independent of the atmospheric depolarisation, in contrast to the other calibrators. Plastic sheet filters can easily be cut to be used in a rotation holder as in Fig. 6, so that the filter can be automatically rotated to $\Delta 90^\circ$ positions and out of the optical path for standard measurements. Large acceptance angles of linearly polarising sheet filters allows the mounting close to the telescope focus where we have some free space and the filter diameter and mechanical mounting can be small due to the small beam diameter. However, it should be considered that the direction of the polarising structure of a sheet filter is not necessarily constant over the whole sheet, which is usually not specified by the manufacturers and should be inquired before the purchase.

$\lambda/4$ plates and circular polariser made of sheet films have similar constraints. Furthermore, the $\Delta 90^\circ$ -calibration doesn't work with a $\lambda/4$ plate, because the $\pm 45^\circ$ errors don't compensate (Eqs. (154), (164)), but in exchange we can determine with it the amount of circular polarisation (Eqs. (158) and (168)). In contrast to that, the ideal circular polariser calibration does not depend on the rotation error ε and the input light polarisation at all and doesn't need a $\Delta 90^\circ$ -calibration, but inherent errors of a real circular polariser, which usually are not sufficiently specified by manufacturers, would be difficult to assess, and the resulting error equations are complex.

While all optical calibrators exhibit wavelength dependency and have the disadvantage of possible inhomogeneities over the surface and other optical errors as inaccurate phase shift or cross-talk, which would add a system depolarisation, the only possible error source of the mechanical rotation calibrator (Sect. 7) is the accuracy of the rotation itself. Although more bulky, it is the most reliable calibrator if used with a cleaned analyser and accurate $\Delta 90^\circ$ rotation (Eq. (105)). It is independent of wavelength, has no internal uncertainties, and is insensitive to temporal changes and degradation.

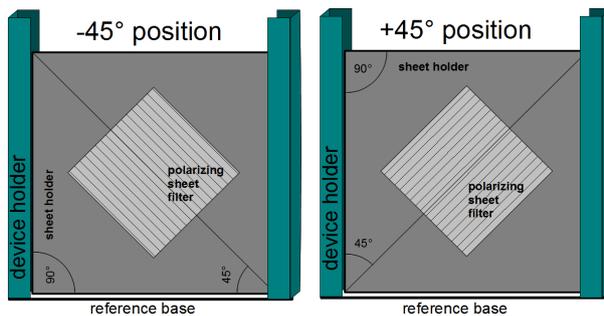


Figure 5: Simple holder for sheet filters (linear polariser or $\lambda/4$ plate) with accurate positioning for the $\Delta 90^\circ$ -calibration.

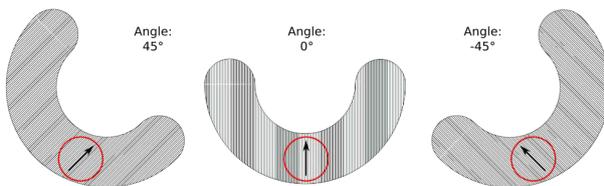


Figure 6: Linearly polarising sheet filter cutout for use in a rotation mount. The optical axis of the filtered light beam is in the centre of the red circle. Reproduced with permission from Kölbl (2010).

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16 Appendix

App. A Acronyms and shortcuts

a	Polarisation parameter of the atmospheric volume; see Eq. 9
a_L	Polarisation parameter of the light beam leaving the laser
a'	$a' = aa_L$, combined laser-atmosphere polarisation parameter
α	Rotation of the plane of horizontal-linear polarisation of the laser around the z-axis (laser rotation)
β	Rotation of the emitter optics around the z-axis
γ	Rotation of the receiver optics around the z-axis
$c_{2\varepsilon}$	$\cos(2\varepsilon)$
δ	Volume linear depolarisation ratio of the atmospheric scattering volume; see Eq. (12)
δ^*	Calibrated signal ratio, but including cross talk and alignment errors
D	Diattenuation parameter (see Sect. 2.2)
ε	error angle of the $\Delta 90$ -calibration setup
$\eta_{T,R}$	Electronic amplification of individual transmitted/reflected channels
η	$\eta = \eta_R T_R / \eta_T T_T$ calibration factor including only the electronic amplification and the optical diattenuation of the polarising beam-splitter
η^*	Gain ratio i.e. the measured, apparent calibration factor η^* of the polarisation channels, i.e. the calibration factor η including the cross talk from optics before the polarising beam-splitter and from system alignment errors
$\eta_{\Delta 90}^*$	$\Delta 90$ -gain ratio $\eta_{\Delta 90}^* \equiv \sqrt{\eta^*(+45^\circ + \varepsilon)\eta^*(-45^\circ + \varepsilon)}$; measured, apparent calibration factor retrieved with the $\Delta 90$ -calibration method
I	Power/flux of the light beam [watt/lumen] (colloquially: intensity)
\mathbf{I}	Stokes vector of the light beam
LDR	Linear depolarisation ratio = δ
\mathbf{F}	Müller matrix of the atmospheric scattering volume in backscattering direction
F_{ij}	Element ij of \mathbf{F}
G_S, H_S	Cross-talk parameters (Sect. 4.1)
K	Correction of the measured gain ratio η^* to get the calibration factor η (Sect. 5)
$\mathbf{M}_S, \mathbf{M}_{T,R}$	Müller matrix of the polarising beam-splitter S , e.g a polarising beam-splitter cube, in the transmission T and reflection R path.
PBS	Polarising beam-splitter
T_S	Transmission of matrix \mathbf{M}_S for unpolarised light (alias average transmission)
T^p, T^s, R^p, R^s	Intensity transmission and reflection coefficients of the polarising beam-splitter for parallel p and perpendicular s linearly polarised light with respect to the plane of incidence.
Z_o	$Z_o = \sqrt{1 - D_o^2}$
W_o	$W_o = 1 - Z_o c_o = 1 - c_o \sqrt{1 - D_o^2}$
$s_{2\varepsilon}$	$\sin(2\varepsilon)$
s_o, c_o	$\sin(\Delta_o), \cos(\Delta_o)$
Δ	Retardation = differential phase shift of the p and s polarised light: $\varphi^p - \varphi^s$
φ^p, φ^s	Phase shift of the p and s polarised light
ψ	Rotation of the calibrator around the z-axis
ϕ	General rotation around z-axis

$\langle \mid$	First row vector of a matrix; analyser vector; bra-vector.
$\mid \rangle$	Stokes vector of the input; always a column vector; ket-vector.

Setup parameters:

h	Binary operator to select either manual rotation (h = +1) or rotation by means of a $\lambda/2$ plate (h = -1)
x, z	Binary operators to select calibration angles of $+45^\circ$ (x, z = +1) or -45° (x, z = -1)
y	Binary operator to select standard measurement setup angles of $+0^\circ$ (y = +1) or $+90^\circ$ (y = -1)

App. B The <bra|ket> notation

Superscript T means the transposition of a row vector to a column vector and vice versa, while the $|\text{ket}\rangle$ and $\langle \text{bra}|$ vector symbols always stand for a column vector and row vector, respectively. That means:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (a \ b \ c \ d)^T = |a \ b \ c \ d\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad (\text{B.1})$$

are forms of column vectors, and

$$(a \ b \ c \ d) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}^T = \langle a \ b \ c \ d| = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad (\text{B.2})$$

are forms of row vectors.

App. C The calibration equation

The general equation for the calibration signals Eq. (81) can be written similar to Kaul et al. (2004) using general expressions for the analyser row vector $\langle \mathbf{A}_S|$ (see App. D) and for the input Stokes vector $|\mathbf{I}_{in}\rangle$ (see App. E)) as in Eq. (C.1), irrespective of the actual position of the calibrator.

$$\frac{I_S}{\eta_S} = \langle \mathbf{A}_S | \mathbf{C}(x45^\circ + \varepsilon) | \mathbf{I}_{in} \rangle = \frac{I_{in}}{\eta_S T_S} \begin{pmatrix} A_S^1 \\ A_S^2 \\ A_S^3 \\ A_S^4 \end{pmatrix} \mathbf{C}(x45^\circ + \varepsilon) \begin{pmatrix} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{pmatrix} \quad (\text{C.1})$$

For certain setups the fully expanded equations are very complex. But sometimes slightly expanded versions are sufficient to achieve significant insights. Demerging the $(\pm 45^\circ + \varepsilon)$ rotation from the calibrator, as in Eq. (C.2), or just the ε – rotations, as in Eq. (C.3), and applying the appropriate parts to the analyser and to the input Stokes vector can help to show general relations. For this purpose we define the rotated analyser vector $\langle \mathbf{A}_{S,\varepsilon} |$ and the rotated input Stokes vector $|\mathbf{I}_{in,\varepsilon}\rangle$ as shown in Eq. (C.3).

$$\begin{aligned}
I_S &= \eta_S \langle \mathbf{A}_S | \mathbf{C}(x45^\circ + \varepsilon) | \mathbf{I}_{in} \rangle = \\
&= \eta_S \langle \mathbf{A}_S | \mathbf{R}(x45^\circ + \varepsilon) \mathbf{C}(0) \mathbf{R}(-x45^\circ - \varepsilon) | \mathbf{I}_{in} \rangle \Rightarrow \\
\frac{I_S}{\eta_S T_S I_{in}} &= \left\langle \begin{array}{c} A_S^1 \\ A_S^2 \\ A_S^3 \\ A_S^4 \end{array} \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -x s_{2\varepsilon} & -x c_{2\varepsilon} & 0 \\ 0 & x c_{2\varepsilon} & -x s_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right. \mathbf{C}(0) \left. \begin{array}{c} 1 & 0 & 0 & 0 \\ 0 & -x s_{2\varepsilon} & x c_{2\varepsilon} & 0 \\ 0 & -x c_{2\varepsilon} & -x s_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| \begin{array}{c} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{array} \right\rangle = \\
&= \left\langle \begin{array}{c} A_S^1 \\ x(c_{2\varepsilon} A_S^3 - s_{2\varepsilon} A_S^2) \\ -x(c_{2\varepsilon} A_S^2 + s_{2\varepsilon} A_S^3) \\ A_S^4 \end{array} \left| \mathbf{C}(0) \right. \begin{array}{c} i_{in} \\ x(c_{2\varepsilon} u_{in} - s_{2\varepsilon} q_{in}) \\ -x(c_{2\varepsilon} q_{in} + s_{2\varepsilon} u_{in}) \\ v_{in} \end{array} \right\rangle = \left\langle \begin{array}{c} A_S^1 \\ x A_{S,\varepsilon}^3 \\ -x A_{S,\varepsilon}^2 \\ A_S^4 \end{array} \left| \mathbf{C}(0) \right. \begin{array}{c} i_{in} \\ x u_{in,\varepsilon} \\ -x q_{in,\varepsilon} \\ v_{in} \end{array} \right\rangle
\end{aligned} \tag{C.2}$$

$$\begin{aligned}
I_S &= \eta_S \langle \mathbf{A}_S | \mathbf{C}(x45^\circ + \varepsilon) | \mathbf{I}_{in} \rangle = \\
&= \eta_S \langle \mathbf{A}_S | \mathbf{R}(+\varepsilon) \mathbf{C}(x45^\circ) \mathbf{R}(-\varepsilon) | \mathbf{I}_{in} \rangle = \eta_S \langle \mathbf{A}_{S,\varepsilon} | \mathbf{C}(x45^\circ) | \mathbf{I}_{in,\varepsilon} \rangle \Rightarrow \\
\frac{I_S}{\eta_S T_S I_{in}} &= \left\langle \begin{array}{c} A_S^1 \\ A_S^2 \\ A_S^3 \\ A_S^4 \end{array} \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & c_{2\varepsilon} & -s_{2\varepsilon} & 0 \\ 0 & s_{2\varepsilon} & c_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right. \mathbf{C}(x45^\circ) \left. \begin{array}{c} 1 & 0 & 0 & 0 \\ 0 & c_{2\varepsilon} & s_{2\varepsilon} & 0 \\ 0 & -s_{2\varepsilon} & c_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| \begin{array}{c} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{array} \right\rangle = \\
&= \left\langle \begin{array}{c} A_S^1 \\ c_{2\varepsilon} A_S^2 + s_{2\varepsilon} A_S^3 \\ c_{2\varepsilon} A_S^3 - s_{2\varepsilon} A_S^2 \\ A_S^4 \end{array} \left| \mathbf{C}(x45^\circ) \right. \begin{array}{c} i_{in} \\ c_{2\varepsilon} q_{in} + s_{2\varepsilon} u_{in} \\ c_{2\varepsilon} u_{in} - s_{2\varepsilon} q_{in} \\ v_{in} \end{array} \right\rangle = \left\langle \begin{array}{c} A_S^1 \\ A_{S,\varepsilon}^2 \\ A_{S,\varepsilon}^3 \\ A_S^4 \end{array} \left| \mathbf{C}(x45^\circ) \right. \begin{array}{c} i_{in} \\ q_{in,\varepsilon} \\ u_{in,\varepsilon} \\ v_{in} \end{array} \right\rangle
\end{aligned} \tag{C.3}$$

Note the exchange of places of $A_{S,\varepsilon}^2$ and $A_{S,\varepsilon}^3$ and of $q_{in,\varepsilon}$ and $u_{in,\varepsilon}$ between Eqs. (C.2) and (C.3).

App. C.1 Calibration with a rotator

From Eqs. (C.1), (C.3), and (S.10.15.2) we get the general calibration signals Eq. (C.4) with analyser vectors $\langle \mathbf{A} |$ from App. D and input Stokes vectors $|\mathbf{I}_{in}\rangle$ from App. E.

$$\begin{aligned}
\frac{I_S}{\eta_S T_S I_{in}} &= \frac{\langle \mathbf{A}_S | \mathbf{M}_{rot}(x45^\circ + \varepsilon, \mathbf{h}) | \mathbf{I}_{in} \rangle}{I_{in}} = \\
&= \left\langle \begin{array}{c} A_S^1 \\ A_S^2 \\ A_S^3 \\ A_S^4 \end{array} \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -x s_{2\varepsilon} & -x h c_{2\varepsilon} & 0 \\ 0 & x c_{2\varepsilon} & -x h s_{2\varepsilon} & 0 \\ 0 & 0 & 0 & h \end{array} \right. \begin{array}{c} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{array} \right\rangle = \\
&= A_S^1 i_{in} + A_S^4 h v_{in} - x \left[(s_{2\varepsilon} A_S^2 - c_{2\varepsilon} A_S^3) q_{in} + (c_{2\varepsilon} A_S^2 + s_{2\varepsilon} A_S^3) h u_{in} \right] = \\
&= A_S^1 i_{in} + A_S^4 h v_{in} + x \left[A_{S,\varepsilon}^3 q_{in} - A_{S,\varepsilon}^2 h u_{in} \right]
\end{aligned} \tag{C.4}$$

App. C.2 Calibration with a linear polariser

From Eq. (C.3) and (S.10.7.1) we get the general calibration signals Eq. (C.5) with analyser vectors $\langle \mathbf{A} |$ from App. D and input Stokes $|I_{in}\rangle$ vectors from App. E. With an ideal linear polariser Eq. (C.5) reduces to Eq. (C.6).

$$\begin{aligned}
 I_S &= \eta_S \langle \mathbf{A}_S | \mathbf{M}_P(x45^\circ + \varepsilon) | I_{in} \rangle = \eta_S \langle \mathbf{A}_S | \mathbf{R}(+\varepsilon) \mathbf{M}_P(x45^\circ) \mathbf{R}(-\varepsilon) | I_{in} \rangle \Rightarrow \\
 \frac{I_S}{\eta_S T_S T_P I_{in}} &= \left\langle A_S^1 \left| \begin{pmatrix} 1 & 0 & xD_P & 0 \\ 0 & Z_P c_P & 0 & -xZ_P s_P \\ xD_P & 0 & 1 & 0 \\ 0 & xZ_P s_P & 0 & Z_P c_P \end{pmatrix} \right| \begin{matrix} i_{in} \\ q_{in,\varepsilon} \\ u_{in,\varepsilon} \\ v_{in} \end{matrix} \right\rangle = \\
 &= \left\{ A_S^1 i_{in} + A_{S,\varepsilon}^3 u_{in,\varepsilon} + Z_P c_P (A_{S,\varepsilon}^2 q_{in,\varepsilon} + A_S^4 v_{in}) + \right. \\
 &\quad \left. + x [D_P (A_S^1 u_{in,\varepsilon} + A_{S,\varepsilon}^3 i_{in}) - Z_P s_P (A_{S,\varepsilon}^2 v_{in} - A_S^4 q_{in,\varepsilon})] \right\} \\
 D_P &= 1, Z_P = 0 \Rightarrow \\
 \frac{I_S}{\eta_S T_S T_P I_{in}} &= A_S^1 i_{in} + A_{S,\varepsilon}^3 u_{in,\varepsilon} + x (A_S^1 u_{in,\varepsilon} + A_{S,\varepsilon}^3 i_{in}) = (A_S^1 + x A_{S,\varepsilon}^3) (i_{in} + x u_{in,\varepsilon}) \quad (C.6)
 \end{aligned}$$

App. C.3 Calibration with a $\lambda/4$ plate (QWP)

From Eq. (C.2) and Eq.(S.10.11.1) for the $\lambda/4$ -plate with retardation error ω as in Eq. (C.7) we get the general calibration signals Eq. (C.8) with an analyser vectors $\langle \mathbf{A} |$ from App. D and input Stokes vectors $|I_{in}\rangle$ from App. E.

$$\Delta_{QW} = 90^\circ + \omega \Rightarrow c_{QW} = -s_\omega, \quad s_{QW} = c_\omega \quad (C.7)$$

$$\begin{aligned}
 \frac{I_S}{\eta_S T_S T_{QW} I_{in}} &= \frac{\langle \mathbf{A}_S | \mathbf{M}_{QW}(x45^\circ + \varepsilon, \omega) | I_{in} \rangle}{T_S T_{QW} I_{in}} = \\
 &= \frac{\langle \mathbf{A}_S | \mathbf{R}(x45^\circ + \varepsilon) \mathbf{M}_{QW}(0, \omega) \mathbf{R}(-x45^\circ - \varepsilon) | I_{in} \rangle}{T_S T_{QW} I_{in}} = \\
 &= \left\langle A_S^1 \left| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -s_\omega & c_\omega \\ 0 & 0 & -c_\omega & -s_\omega \end{pmatrix} \right| \begin{matrix} i_{in} \\ x u_{in,\varepsilon} \\ -x q_{in,\varepsilon} \\ v_{in} \end{matrix} \right\rangle = \\
 &= A_S^1 i_{in} + A_{S,\varepsilon}^3 u_{in,\varepsilon} - s_\omega (A_{S,\varepsilon}^2 q_{in,\varepsilon} + A_S^4 v_{in}) - x c_\omega (-A_S^4 q_{in,\varepsilon} + A_{S,\varepsilon}^2 v_{in}) \\
 \omega &= 0 \Rightarrow \\
 \frac{I_S}{\eta_S T_S T_{QW} I_{in}} &= \frac{\langle \mathbf{A}_S | \mathbf{M}_{QW}(x45^\circ + \varepsilon, 0) | I_{in} \rangle}{T_S T_{QW} I_{in}} = A_S^1 i_{in} + A_{S,\varepsilon}^3 u_{in,\varepsilon} - x (-A_S^4 q_{in,\varepsilon} + A_{S,\varepsilon}^2 v_{in}) = \\
 &= A_S^1 i_{in} - (s_{2\varepsilon} A_S^2 - c_{2\varepsilon} A_S^3) (s_{2\varepsilon} q_{in} - c_{2\varepsilon} u_{in}) - x [-A_S^4 (c_{2\varepsilon} q_{in} + s_{2\varepsilon} u_{in}) + (c_{2\varepsilon} A_S^2 + s_{2\varepsilon} A_S^3) v_{in}] \quad (C.9)
 \end{aligned}$$

App. C.4 Calibration with a circular polariser (CP)

From Eq. (C.2) for a circular polariser composed of a linear polariser and a $\lambda/4$ -plate with retardation error ω as in Eq. (C.7) we get the general calibration signals Eq. (C.10) with

analyser vectors $\langle \mathbf{A} |$ from App. D and input Stokes vectors $| \mathbf{I}_{in} \rangle$ from App. E. Note that $z = \pm 1$ discerns between a right and left circular polariser, and $x = \pm 1$ between the $\pm 45^\circ$ orientations of the whole circular polariser. With an ideal linear polariser this quite complex equation reduces to Eq. (C.11), with an ideal QWP without retardation error to Eq. (C.12), and to Eq. (C.13) with both constraints, i.e. for an ideal circular polariser. Since only the terms with an x in Eqs. (C.11) to (C.13) are compensated by means of the $\Delta 90$ -calibration, neither of the two constraints alone is sufficient to reduce the uncertainty.

$$\begin{aligned}
\frac{I_S}{\eta_S T_S T_{CP} I_{in}} &= \frac{\langle \mathbf{A}_S | \mathbf{M}_{CP}(z, x 45^\circ + \varepsilon) | \mathbf{I}_{in} \rangle}{T_S T_{CP} I_{in}} = \\
&= \frac{\langle \mathbf{A}_S \mathbf{R}(x 45^\circ + \varepsilon) | \mathbf{M}_{QW}(z 45^\circ, \omega) \mathbf{M}_P | \mathbf{R}(-x 45^\circ - \varepsilon) \mathbf{I}_{in} \rangle}{T_S T_{QW} T_P I_{in}} = \\
&= \left\langle \begin{array}{c} A_S^1 \\ x A_{S,\varepsilon}^3 \\ -x A_{S,\varepsilon}^2 \\ A_S^4 \end{array} \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -s_\omega & 0 & -z c_\omega \\ 0 & 0 & 1 & 0 \\ 0 & z c_\omega & 0 & -s_\omega \end{array} \right| \begin{array}{cccc} 1 & D_P & 0 & 0 \\ D_P & 1 & 0 & 0 \\ 0 & 0 & Z_P c_P & Z_P s_P \\ 0 & 0 & -Z_P s_P & Z_P c_P \end{array} \right| \begin{array}{c} i_{in} \\ x u_{in,\varepsilon} \\ -x q_{in,\varepsilon} \\ v_{in} \end{array} \right\rangle = \quad (C.10) \\
&= \left\{ \begin{array}{l} \left(A_S^1 + A_S^4 z c_\omega D_P \right) i_{in} + \left(A_{S,\varepsilon}^2 Z_P c_P - z A_{S,\varepsilon}^3 c_\omega Z_P s_P \right) q_{in,\varepsilon} - A_{S,\varepsilon}^3 s_\omega u_{in,\varepsilon} - A_S^4 s_\omega Z_P c_P v_{in} + \\ + x \left[\begin{array}{l} -A_{S,\varepsilon}^3 s_\omega D_P i_{in} - A_S^4 s_\omega Z_P s_P q_{in,\varepsilon} + \\ \left(A_S^1 D_P + A_S^4 z c_\omega \right) u_{in,\varepsilon} - \left(A_{S,\varepsilon}^2 s_P + A_{S,\varepsilon}^3 z c_\omega c_P \right) Z_P v_{in} \end{array} \right] \end{array} \right\}
\end{aligned}$$

From Eq.(C.10) we get with different conditions:

$$\begin{aligned}
D_P = 1, Z_P = 0 &\Rightarrow \\
\frac{I_S}{\eta_S T_S T_{CP} I_{in}} &= \left(A_S^1 + z c_\omega A_S^4 - s_\omega x A_{S,\varepsilon}^3 \right) \left(i_{in} + x u_{in,\varepsilon} \right) \quad (C.11)
\end{aligned}$$

$$\begin{aligned}
\omega = 0 &\Rightarrow \\
\frac{I_S}{\eta_S T_S T_{CP} I_{in}} &= \left\{ \begin{array}{l} \left(A_S^1 + A_S^4 z D_P \right) i_{in} + \left(A_{S,\varepsilon}^2 Z_P c_P - z A_{S,\varepsilon}^3 Z_P s_P \right) q_{in,\varepsilon} \\ + x \left[\left(A_S^1 D_P + z A_S^4 \right) u_{in,\varepsilon} - \left(A_{S,\varepsilon}^2 s_P + z A_{S,\varepsilon}^3 c_P \right) Z_P v_{in} \right] \end{array} \right\} \quad (C.12)
\end{aligned}$$

$$\begin{aligned}
\omega = 0, D_P = 1, Z_P = 0 &\Rightarrow \\
\frac{I_S}{\eta_S T_S T_{CP} I_{in}} &= \left(A_S^1 + z A_S^4 \right) \left(i_{in} - x u_{in,\varepsilon} \right) \quad (C.13)
\end{aligned}$$

App. D The analyser row vector $\langle \mathbf{A}_S |$

The general formulation for the Stokes vector of a standard lidar signal \mathbf{I}_S at the detector in the reflected channel, \mathbf{I}_R , and transmitted channel, \mathbf{I}_T , is

$$\mathbf{I}_S = \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L \quad (D.1)$$

Only the first Stokes parameter is directly measured, and therefore we can reduce the complexity of the full matrix equations to an inner product between the analyser row vector $\langle \mathbf{A}_S |$ and the input Stokes column vector \mathbf{I}_{in} similar to Kaul et al. (1992) and Volkov et al. (2015) as in Eq. (D.2):

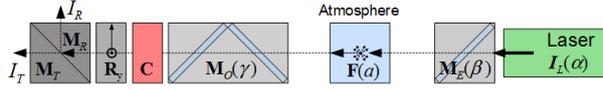
$$I_S = \langle \mathbf{A}_S | \mathbf{I}_{in} \rangle \quad (\text{D.2})$$

In case of a calibration measurement, we place a calibrator with matrix \mathbf{C} between the input Stokes vector and the analyser vector:

$$I_S = \langle \mathbf{A}_S | \mathbf{C} | \mathbf{I}_{in} \rangle \quad (\text{D.3})$$

As calibrators we use a mechanical rotator, a rotation of the plane of polarisation by means of a $\lambda/2$ plate (HWP), a linear polariser, a $\lambda/4$ plate (QWP), and a circular polariser. We can place the calibrator anywhere in the optical setup, with different results. In the following we develop the general expressions of the analyser vector in App. D and of the input Stokes vector in App. E for the different setups.

App. D.1 $\langle \mathbf{A}_S |$ with \mathbf{C} before the polarising beam-splitter

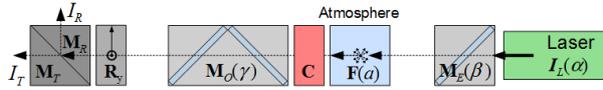


$$I_S = \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{C} \mathbf{M}_O \mathbf{F} I_E \Rightarrow \mathbf{A}_S = \mathbf{M}_S \mathbf{R}_y \quad (\text{D.4})$$

The analyser part consists of a polarising beam-splitter \mathbf{M}_S and an optional 90° rotation of the detector setup \mathbf{R}_y (see Eq.(47))

$$\begin{aligned} \frac{\langle \mathbf{A}_S |}{T_S} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y |}{T_S} = \\ &= \left\langle \begin{pmatrix} 1 & D_S & 0 & 0 \\ D_S & 1 & 0 & 0 \\ 0 & 0 & Z_S c_S & Z_S s_S \\ 0 & 0 & -Z_S s_S & Z_S c_S \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = \left\langle \begin{pmatrix} 1 & y D_S & 0 & 0 \\ D_S & y & 0 & 0 \\ 0 & 0 & y Z_S c_S & Z_S s_S \\ 0 & 0 & -y Z_S s_S & Z_S c_S \end{pmatrix} \right| = \left\langle \begin{pmatrix} 1 \\ y D_S \\ 0 \\ 0 \end{pmatrix} \right| \end{aligned} \quad (\text{D.5})$$

App. D.2 $\langle \mathbf{A}_S |$ with \mathbf{C} before the receiving optics



$$I_S = \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O \mathbf{C} \mathbf{F} I_E \Rightarrow \mathbf{A}_S = \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O \quad (\text{D.6})$$

Using Eq. D.5 we get

$$\begin{aligned} \frac{\langle \mathbf{A}_S(y, \gamma) |}{T_O T_S} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{M}_O(\gamma) |}{T_O T_S} = \\ &= \left\langle \begin{pmatrix} 1 \\ y D_S \\ 0 \\ 0 \end{pmatrix} \left| \begin{pmatrix} 1 & c_{2\gamma} D_O & s_{2\gamma} D_O & 0 \\ c_{2\gamma} D_O & 1 - s_{2\gamma}^2 W_O & s_{2\gamma} c_{2\gamma} W_O & -s_{2\gamma} Z_O s_O \\ s_{2\gamma} D_O & s_{2\gamma} c_{2\gamma} W_O & 1 - c_{2\gamma}^2 W_O & c_{2\gamma} Z_O s_O \\ 0 & s_{2\gamma} Z_O s_O & -c_{2\gamma} Z_O s_O & Z_O c_O \end{pmatrix} \right. \right\rangle = \left\langle \begin{pmatrix} 1 + y c_{2\gamma} D_O D_S \\ c_{2\gamma} D_O + y D_S (1 - s_{2\gamma}^2 W_O) \\ s_{2\gamma} (D_O + y c_{2\gamma} D_S W_O) \\ -y s_{2\gamma} D_S Z_O s_O \end{pmatrix} \right| \end{aligned} \quad (\text{D.7})$$

Simplifications: A rotation γ of a retarding diattenuator \mathbf{M}_O between the calibrator and the polarising beam-splitter \mathbf{M}_S complicates the equations considerably. In case \mathbf{M}_O is not rotated

($\gamma = 0$), the matrices \mathbf{M}_S , the optional 90° rotation \mathbf{R}_y , and \mathbf{M}_O and can be combined to a new polarising beam-splitter module \mathbf{M}_{SyO} according to S.10.10, and all equations developed for the Sect. 7.1 case can be applied in Sect. 7.2. For $\gamma = 0^\circ$ Eq. (D.7) becomes

$$\begin{aligned} \gamma = 0^\circ &\Rightarrow \\ \langle \mathbf{A}_S | (y, 0^\circ) &= \langle \mathbf{M}_S \mathbf{R}_y | \mathbf{M}_O (0^\circ) = T_O T_S \langle 1 + yD_S D_O \quad D_O + yD_S \quad 0 \quad 0 | = \\ &= \langle \mathbf{M}_{SyO} (0^\circ) | = T_{SyO} \langle 1 \quad D_{SyO} \quad 0 \quad 0 | \end{aligned} \quad (\text{D.8})$$

$$\text{with } T_{SyO} = T_O T_S (1 + yD_S D_O) \text{ and } D_{SyO} = \frac{D_O + yD_S}{1 + yD_S D_O} \quad (\text{D.9})$$

With a cleaned analyser we get from Eq. (D.9)

$$\begin{aligned} D_R = -1, D_T = +1 &\Rightarrow \\ D_{SyO} = yD_S, \quad D_{RyO} = -y, \quad D_{TyO} = +y & \\ T_{RyO} = T_O T_R (1 - yD_O), \quad T_{TyO} = T_O T_T (1 + yD_O) & \end{aligned} \quad (\text{D.10})$$

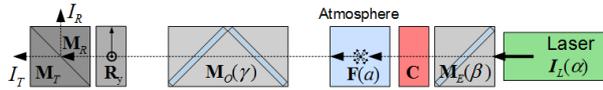
and explicitly with Eqs. (S.10.10.11) and (S.10.10.14)

$$\begin{aligned} D_R = -1, D_T = +1, y = +1 &\Rightarrow \\ T_{R+O} = T_O T_R (1 - D_O) = 0.5 T_R^s k_1 T_O^s, \quad D_{R+O} = -1 & \\ T_{T+O} = T_O T_T (1 + D_O) = 0.5 T_T^p k_1 T_O^p, \quad D_{T+O} = +1 & \\ D_R = -1, D_T = +1, y = -1 &\Rightarrow \\ T_{R-O} = T_O T_R (1 + D_O) = 0.5 T_R^s k_1 T_O^p, \quad D_{R-O} = +1 & \\ T_{T-O} = T_O T_T (1 - D_O) = 0.5 T_T^p k_1 T_O^s, \quad D_{T-O} = -1 & \end{aligned} \quad (\text{D.11})$$

See also Suppl. S.10.10 and S.6 .

Only few special cases with rotated \mathbf{M}_O ($\gamma \neq 0$) (see Eq. (S.5.1.4)) are discussed additionally.

App. D.3 $\langle \mathbf{A}_S |$ with C behind the emitter optics



$$\mathbf{I}_S = \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O \mathbf{F} \mathbf{C} \mathbf{I}_E \Rightarrow \mathbf{A}_S = \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O \mathbf{F} \mathbf{C} \text{ and } \mathbf{I}_{in} = \mathbf{I}_E \quad (\text{D.12})$$

The additional effect of the atmospheric depolarisation, $\mathbf{F}(a)$, on the analyser Eq. (D.7) is

$$\begin{aligned} \frac{\langle \mathbf{A}_S |}{T_O T_S F_{11}} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O (\gamma) | \mathbf{F}(a)}{T_O T_S F_{11}} = \\ &= \left\langle \begin{array}{l} 1 + y c_{2\gamma} D_S D_O \\ c_{2\gamma} D_O + y D_S (1 - s_{2\gamma}^2 W_O) \\ s_{2\gamma} (D_O + y c_{2\gamma} D_S W_O) \\ -y s_{2\gamma} D_S Z_O S_O \end{array} \right| \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1 - 2a \end{array} \right) = \left\langle \begin{array}{l} 1 + y c_{2\gamma} D_S D_O \\ a [c_{2\gamma} D_O + y D_S (1 - s_{2\gamma}^2 W_O)] \\ -a s_{2\gamma} (D_O + y c_{2\gamma} D_S W_O) \\ -(1 - 2a) y s_{2\gamma} D_S Z_O S_O \end{array} \right| \end{aligned} \quad (\text{D.13})$$

Without receiver optics rotation \mathbf{M}_O ($\gamma = 0^\circ$) we get with Eq. (D.8) ff.

$$\langle \mathbf{A}_S | = \langle \mathbf{M}_{SyO}(0^\circ) | \mathbf{F}(a) = T_{SyO} \langle 1 \quad aD_{SyO} \quad 0 \quad 0 | \quad (\text{D.14})$$

App. E The input Stokes vector I_{in}

The formulation for the most general input Stokes vector I_{in} into the analyser part \mathbf{A}_S is

$$\mathbf{I}_{in}(\gamma, a, \beta) = \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L \quad (\text{E.1})$$

and assuming a rotated, partly linear polarised laser with polarisation parameter a_L

$$\mathbf{I}_{in}(\gamma, a, \beta, \alpha, a_L) = \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha, a_L) \quad (\text{E.2})$$

In the ideal case the laser has no depolarisation ($a_L = 1$) and is horizontal linearly polarised (see Eq. (E.6)), and the optical elements are not rotated, which results in Eq.(E.3):

$$\begin{aligned} a_L = 1, i_L = q_L = 1, u_L = v_L = 0, \alpha = \beta = \gamma = 0 \Rightarrow \\ \mathbf{I}_{in}(0, 0, 0, 0, 1) = T_O F_{11} T_E I_L (1 + D_E) | 1 + aD_O \quad D_O + a \quad 0 \quad 0 \rangle \end{aligned} \quad (\text{E.3})$$

App. E.1 Laser I_L

We start with the Stokes vector for the laser beam with arbitrary state of polarisation and additionally rotated by angle α around the optical axis (see Eq. S.5.1.1)

$$\mathbf{I}_L(\alpha) = I_L \begin{pmatrix} i_{L,\alpha} \\ q_{L,\alpha} \\ u_{L,\alpha} \\ v_{L,\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\alpha} & -s_{2\alpha} & 0 \\ 0 & s_{2\alpha} & c_{2\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} I_L \begin{pmatrix} i_L \\ q_L \\ u_L \\ v_L \end{pmatrix} = I_L \begin{pmatrix} i_L \\ c_{2\alpha} q_L - s_{2\alpha} u_L \\ s_{2\alpha} q_L + c_{2\alpha} u_L \\ v_L \end{pmatrix} \quad (\text{E.4})$$

The total, linear, and circular degree of polarisation (DOP, D \underline{O} LP, and D \underline{O} CP, respectively) don't change with such a rotation.

- We get for a rotated, horizontal-linear polarised laser

$$\mathbf{I}_L(\alpha) = I_L | 1 \quad c_{2\alpha} \quad s_{2\alpha} \quad 0 \rangle \quad (\text{E.5})$$

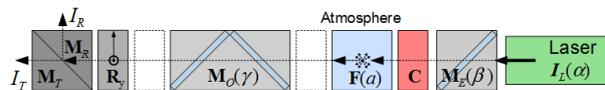
- for a horizontal-linear polarised laser

$$\mathbf{I}_L(0) = I_L | 1 \quad 1 \quad 0 \quad 0 \rangle \quad (\text{E.6})$$

- and for a rotated, linearly polarised laser with polarisation parameter a_L with $\delta_L = (1-a_L)/(1+a_L)$

$$\mathbf{I}_L(\alpha, a_L) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\alpha} & -s_{2\alpha} & 0 \\ 0 & s_{2\alpha} & c_{2\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} I_L \begin{pmatrix} 1 \\ a_L \\ 0 \\ 0 \end{pmatrix} = I_L \begin{pmatrix} 1 \\ c_{2\alpha} a_L \\ s_{2\alpha} a_L \\ 0 \end{pmatrix} \quad (\text{E.7})$$

App. E.2 I_{in} with C behind the emitter optics



$$\begin{aligned} \frac{\mathbf{I}_{in}(\beta, \alpha)}{T_E I_L} &= \frac{\mathbf{I}_E(\beta, \alpha)}{T_E I_L} = \frac{\mathbf{M}_E(\beta) \mathbf{I}_L(\alpha)}{T_E I_L} = \frac{\mathbf{M}_E(\beta)}{T_E I_L} |i_L \ q_L \ u_L \ v_L\rangle = \\ &= |i_{in} \ q_{in} \ u_{in} \ v_{in}\rangle = |i_E \ q_E \ u_E \ v_E\rangle \end{aligned} \quad (\text{E.8})$$

Eq. (E.8) with input I_L from a rotated, linearly polarised laser Eq. (E.4) and with rotated emitter optics Eq. (S.10.4.1) results in Eq. (E.9).

$$\begin{aligned} \frac{\mathbf{I}_{in}(\beta, \alpha)}{T_E I_L} &= \frac{\mathbf{I}_E(\beta, \alpha)}{T_E I_L} = \frac{\mathbf{M}_E(\beta) |\mathbf{I}_L(\alpha)\rangle}{T_E I_L} = |i_E \ q_E \ u_E \ v_E\rangle = \\ &= \left(\begin{array}{cccc} 1 & c_{2\beta} D_E & s_{2\beta} D_E & 0 \\ c_{2\beta} D_E & 1 - s_{2\beta}^2 W_E & s_{2\beta} c_{2\beta} W_E & -s_{2\beta} Z_E S_E \\ s_{2\beta} D_E & s_{2\beta} c_{2\beta} W_E & 1 - c_{2\beta}^2 W_E & c_{2\beta} Z_E S_E \\ 0 & s_{2\beta} Z_E S_E & -c_{2\beta} Z_E S_E & Z_E c_E \end{array} \right) \left| \begin{array}{c} i_L \\ c_{2\alpha} q_L - s_{2\alpha} u_L \\ s_{2\alpha} q_L + c_{2\alpha} u_L \\ v_L \end{array} \right\rangle = \\ &= \left| \begin{array}{c} i_L + D_E (c_{2\alpha-2\beta} q_L - s_{2\alpha-2\beta} u_L) \\ c_{2\beta} D_E i_L + (c_{2\alpha} q_L - s_{2\alpha} u_L) + s_{2\beta} [W_E (s_{2\alpha-2\beta} q_L + c_{2\alpha-2\beta} u_L) - Z_E S_E v_L] \\ s_{2\beta} D_E i_L + (s_{2\alpha} q_L + c_{2\alpha} u_L) - c_{2\beta} [W_E (s_{2\alpha-2\beta} q_L + c_{2\alpha-2\beta} u_L) - Z_E S_E v_L] \\ -Z_E S_E (s_{2\alpha-2\beta} q_L + c_{2\alpha-2\beta} u_L) + Z_E c_E v_L \end{array} \right\rangle \end{aligned} \quad (\text{E.9})$$

• Special cases: Eq. (E.9) without rotation of the emitter optics with respect to the plane of polarisation of the laser

$$\alpha = \beta \Rightarrow$$

$$\begin{aligned} \frac{\mathbf{I}_{in}(\alpha, \alpha)}{T_E I_L} &= \frac{\mathbf{I}_E(\alpha, \alpha)}{T_E I_L} = \left| \begin{array}{c} i_E \\ q_E \\ u_E \\ v_E \end{array} \right\rangle = \left| \begin{array}{c} i_L + D_E q_L \\ c_{2\alpha} D_E i_L + (c_{2\alpha} q_L - s_{2\alpha} u_L) + s_{2\alpha} [W_E u_L - Z_E S_E v_L] \\ s_{2\alpha} D_E i_L + (s_{2\alpha} q_L + c_{2\alpha} u_L) - c_{2\alpha} [W_E u_L - Z_E S_E v_L] \\ -Z_E S_E u_L + Z_E c_E v_L \end{array} \right\rangle = \\ &= \left| \begin{array}{c} i_L + D_E q_L \\ c_{2\alpha} (D_E i_L + q_L) - s_{2\alpha} Z_E (c_E u_L + s_E v_L) \\ s_{2\alpha} (D_E i_L + q_L) + c_{2\alpha} Z_E (c_E u_L + s_E v_L) \\ -Z_E (s_E u_L - c_E v_L) \end{array} \right\rangle \end{aligned} \quad (\text{E.10})$$

• Eq. (E.9) without laser and emitter optics rotation

$$\alpha = \beta = 0 \Rightarrow$$

$$\frac{\mathbf{I}_{in}(0,0)}{T_E I_L} = \frac{\mathbf{I}_E(0,0)}{T_E I_L} = \frac{\mathbf{M}_E(0) |\mathbf{I}_L(0)\rangle}{T_E I_L} = \left| \begin{array}{c} i_E \\ q_E \\ u_E \\ v_E \end{array} \right\rangle = \left| \begin{array}{c} i_L + D_E q_L \\ D_E i_L + q_L \\ Z_E (c_E u_L + s_E v_L) \\ Z_E (-s_E u_L + c_E v_L) \end{array} \right\rangle \quad (\text{E.11})$$

• Eq. (E.9) with rotated, horizontal-linearly polarised laser with rotated emitter optics

$$\mathbf{I}_L = I_L |1 \ 1 \ 0 \ 0\rangle \Rightarrow$$

$$\frac{\mathbf{I}_{in}(\beta, \alpha)}{T_E I_L} = \frac{\mathbf{I}_E(\beta, \alpha)}{T_E I_L} = \frac{\mathbf{M}_E(\beta) |\mathbf{I}_L(\alpha)\rangle}{T_E I_L} = \begin{pmatrix} i_E \\ q_E \\ u_E \\ v_E \end{pmatrix} = \begin{pmatrix} 1 + D_E c_{2\alpha-2\beta} \\ c_{2\alpha} + c_{2\beta} D_E + s_{2\beta} W_E s_{2\alpha-2\beta} \\ s_{2\alpha} + s_{2\beta} D_E - c_{2\beta} W_E s_{2\alpha-2\beta} \\ -Z_E s_E s_{2\alpha-2\beta} \end{pmatrix} \quad (\text{E.12})$$

• Eq. (E.9) with rotated, linearly polarised laser without emitter optics rotation

$$\alpha = \beta \wedge \mathbf{I}_L = I_L |1 \ 1 \ 0 \ 0\rangle \Rightarrow$$

$$\mathbf{I}_{in}(\alpha, \alpha) = T_E I_L |i_E \ q_E \ u_E \ v_E\rangle = T_E I_L (1 + D_E) |1 \ c_{2\alpha} \ s_{2\alpha} \ 0\rangle \quad (\text{E.13})$$

• Rotated, elliptically polarised light behind the emitter optics with.

$$\mathbf{I}_{in} = \mathbf{I}_E = T_E I_L |i_E \ q_E \ u_E \ v_E\rangle = T_E I_L |1 \ bc_{2\alpha} \ bs_{2\alpha} \ v_E\rangle \quad (\text{E.14})$$

with the degree of polarisation $DOP_E = 1$ and the degree of linear polarisation $DOLP_E = b$

$$DOP_E = \sqrt{q_E^2 + u_E^2 + v_E^2} = \sqrt{b^2 + v_E^2} = 1 \Rightarrow v_E = \sqrt{1 - b^2} \quad (\text{E.15})$$

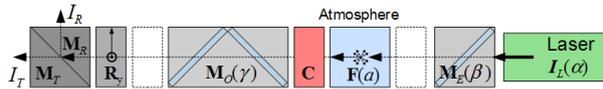
$$\mathbf{I}_{in} = \mathbf{I}_E = T_E I_L |i_E \ q_E \ u_E \ v_E\rangle = T_E I_L |1 \ bc_{2\alpha} \ bs_{2\alpha} \ \sqrt{1 - b^2}\rangle \quad (\text{E.16})$$

• Rotated, linearly polarised laser with linear polarisation parameter a_L with rotated emitter optics: Laser Stokes vector Eq.(E.7) and rotated diattenuator Eq.(S.10.4.1)

$$\mathbf{I}_L = I_L |1 \ c_{2\alpha} a_L \ s_{2\alpha} a_L \ 0\rangle \Rightarrow$$

$$\frac{\mathbf{I}_{in}}{T_E I_L} = \frac{\mathbf{I}_E}{T_E I_L} = \frac{\mathbf{M}_E(\beta) |\mathbf{I}_L(\alpha, a_L)\rangle}{T_E I_L} = |i_E \ q_E \ u_E \ v_E\rangle = \begin{pmatrix} 1 & c_{2\beta} D_E & s_{2\beta} D_E & 0 \\ c_{2\beta} D_E & 1 - s_{2\beta}^2 W_E & s_{2\beta} c_{2\beta} W_E & -s_{2\beta} Z_E s_E \\ s_{2\beta} D_E & s_{2\beta} c_{2\beta} W_E & 1 - c_{2\beta}^2 W_E & c_{2\beta} Z_E s_E \\ 0 & s_{2\beta} Z_E s_E & -c_{2\beta} Z_E s_E & Z_E c_E \end{pmatrix} \begin{pmatrix} 1 \\ c_{2\alpha} a_L \\ s_{2\alpha} a_L \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + a_L D_E c_{2\alpha-2\beta} \\ c_{2\beta} D_E + a_L (c_{2\alpha} + s_{2\beta} W_E s_{2\alpha-2\beta}) \\ s_{2\beta} D_E + a_L (s_{2\alpha} - c_{2\beta} W_E s_{2\alpha-2\beta}) \\ -a_L Z_E s_E s_{2\alpha-2\beta} \end{pmatrix} \quad (\text{E.17})$$

App. E.3 \mathbf{I}_{in} with C before the receiver optics



General input Stokes \mathbf{I}_{in} vector with atmospheric backscatter.

$$\mathbf{I}_S = \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O \mathbf{C} \mathbf{F} \mathbf{I}_E \Rightarrow \mathbf{I}_{in} = \mathbf{F} \mathbf{I}_E \quad (\text{E.18})$$

With atmospheric depolarisation from Eq. (S.3.1) and an emitter beam \mathbf{I}_E from App. E.2:

$$\mathbf{I}_{in}(a) = |\mathbf{F}(a) \mathbf{I}_E\rangle = F_{11} T_E I_L |i_E \ aq_E \ -au_E \ (1 - 2a)v_E\rangle \quad (\text{E.19})$$

• Special cases: Eq. (E.19) becomes Eq. (E.20) with a rotated linearly polarised laser with linear polarisation parameter a_L , with rotated emitter optics, and atmospheric backscatter, i.e. Eq. (E.17). Note, that without laser depolarisation $a_L = 1$.

$$\frac{\mathbf{I}_{in}(a, \beta, \alpha, a_L)}{F_{11}T_E I_L} = \frac{\mathbf{F}(a) |\mathbf{M}_E(\beta) \mathbf{I}_L(\alpha, a_L)\rangle}{F_{11}T_E I_L} =$$

$$= \left\langle \begin{array}{c} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{array} \right\rangle = \left\langle \begin{array}{c} i_E \\ aq_E \\ -au_E \\ (1-2a)v_E \end{array} \right\rangle = \left\langle \begin{array}{c} 1 + a_L D_E c_{2\alpha-2\beta} \\ a [c_{2\beta} D_E + a_L (c_{2\alpha} + s_{2\beta} W_E s_{2\alpha-2\beta})] \\ -a [s_{2\beta} D_E + a_L (s_{2\alpha} - c_{2\beta} W_E s_{2\alpha-2\beta})] \\ -(1-2a) a_L Z_E s_E s_{2\alpha-2\beta} \end{array} \right\rangle \quad (\text{E.20})$$

• Eq. (E.20) without rotation errors becomes Eq. (E.21), and additionally without laser depolarisation, i.e. $a_L = 1$, Eq. (E.22).

$$\alpha = \beta = 0 \Rightarrow$$

$$\frac{\mathbf{I}_{in}(a, 0, 0, a_L)}{F_{11}T_E I_L} = \frac{\mathbf{F}(a) |\mathbf{M}_E(0) \mathbf{I}_L(0, a_L)\rangle}{F_{11}T_E I_L} = |1 + a_L D_E \quad a D_E + a a_L \quad 0 \quad 0\rangle \quad (\text{E.21})$$

$$\mathbf{I}_{in}(a, 0, 0, 0) = F_{11}T_E I_L (1 + D_E) |1 \quad a \quad 0 \quad 0\rangle \quad (\text{E.22})$$

• Eq. (E.20) without emitter optics becomes Eq. (E.23).

$$[D_E = 0 \Rightarrow Z_E = 1, s_E = 0 \Rightarrow c_E = 1 \Rightarrow W_E = 0] \Rightarrow$$

$$\frac{\mathbf{I}_{in}(a, \alpha, a_L)}{F_{11}I_L} = \frac{\mathbf{F}(a) |\mathbf{I}_L(\alpha, a_L)\rangle}{F_{11}I_L} = |1 \quad a a_L c_{2\alpha} \quad -a a_L s_{2\alpha} \quad 0\rangle \quad (\text{E.23})$$

Note it is impossible to combine $a' = a a_L$ if emitter optics \mathbf{M}_E with diattenuation parameter $D_E \neq 0$ or retardation (i.e. $Z_E \neq 0$ and $s_E \neq 0$) are between the laser and the atmosphere \mathbf{F} , even if there are no angular misalignments α and β in the emitter, which means that the atmospheric depolarisation cannot be retrieved without detailed knowledge of the emitter optics parameters and alignment errors.

• Eq. (E.20) without emitter optics \mathbf{M}_E and without laser depolarisation becomes Eq. (E.24).

$$a_L = 1, [D_E = 0 \Rightarrow Z_E = 1, s_E = 0 \Rightarrow c_E = 1 \Rightarrow W_E = 0] \Rightarrow$$

$$\frac{\mathbf{I}_{in}(a, \alpha)}{I_{in}} = \frac{\mathbf{F}(a) |\mathbf{I}_L(\alpha)\rangle}{F_{11}I_L} = |1 \quad a c_{2\alpha} \quad -a s_{2\alpha} \quad 0\rangle \quad (\text{E.24})$$

• Eq. (E.19) with \mathbf{I}_E from Eq. (E.14), i.e. with rotated, elliptically polarised light behind the emitter optics

$$\frac{\mathbf{I}_{in}(a, b, \alpha)}{I_{in}} = \frac{\mathbf{F}(a) \mathbf{I}_E}{F_{11}T_E I_L} = |i_E \quad aq_E \quad -au_E \quad (1-2a)v_E\rangle =$$

$$= |1 \quad abc_{2\alpha} \quad -abs_{2\alpha} \quad (1-2a)\sqrt{1-b^2}\rangle \quad (\text{E.25})$$

• Including the calibrator rotation $R(\varepsilon)$ in I_{in} in Eq. (E.19) with Eq. (S.10.15.1) gives Eq. (E.26), and with an elliptically polarised laser of Eq. (E.16) we get Eq. (E.27), which results without emitter optics and horizontal-linear polarised laser light ($b = 1$) in Eq. (E.28).

$$\frac{\mathbf{I}_{in,\varepsilon}(\varepsilon, h, a)}{I_{in}} = \frac{|\mathbf{R}(\varepsilon)\mathbf{M}_h\mathbf{F}(a)\mathbf{I}_E\rangle}{T_{rot}F_{11}T_E I_L} =$$

$$= \left| \begin{array}{c} i_{in,\varepsilon} \\ q_{in,\varepsilon} \\ u_{in,\varepsilon} \\ v_{in,\varepsilon} \end{array} \right\rangle = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & c_{2\varepsilon} & -hs_{2\varepsilon} & 0 \\ 0 & s_{2\varepsilon} & hc_{2\varepsilon} & 0 \\ 0 & 0 & 0 & h \end{array} \right) \left| \begin{array}{c} i_E \\ aq_E \\ -au_E \\ (1-2a)v_E \end{array} \right\rangle = \left| \begin{array}{c} i_E \\ a(q_E c_{2\varepsilon} + hu_E s_{2\varepsilon}) \\ a(q_E s_{2\varepsilon} - hu_E c_{2\varepsilon}) \\ (1-2a)hv_E \end{array} \right\rangle \quad (\text{E.26})$$

$$\mathbf{I}_E = T_E I_L |i_E \quad q_E \quad u_E \quad v_E\rangle = T_E I_L |1 \quad bc_{2\alpha} \quad bs_{2\alpha} \quad \sqrt{1-b^2}\rangle \Rightarrow$$

$$\frac{\mathbf{I}_{in,\varepsilon}(\varepsilon, h, a, \alpha, b)}{I_{in}} = \frac{|\mathbf{R}(\varepsilon)\mathbf{M}_h\mathbf{F}(a)\mathbf{I}_E(\alpha, b)\rangle}{T_{rot}F_{11}T_E I_L} =$$

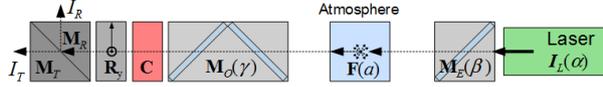
$$= |i_{in,\varepsilon} \quad q_{in,\varepsilon} \quad u_{in,\varepsilon} \quad v_{in,\varepsilon}\rangle = |1 \quad abc_{2\varepsilon-h2\alpha} \quad abs_{2\varepsilon+h2\alpha} \quad (1-2a)h\sqrt{1-b^2}\rangle \quad (\text{E.27})$$

$$\mathbf{M}_E = \text{identity}, b = 1 \Rightarrow$$

$$\frac{\mathbf{I}_{in,\varepsilon}(\varepsilon, h, a, \alpha, b)}{I_{in}} = \frac{|\mathbf{R}(\varepsilon)\mathbf{M}_h\mathbf{F}(a)\mathbf{I}_L(\alpha, b)\rangle}{T_{rot}F_{11}I_L} =$$

$$= |i_{in,\varepsilon} \quad q_{in,\varepsilon} \quad u_{in,\varepsilon} \quad v_{in,\varepsilon}\rangle = |1 \quad ac_{2\varepsilon-h2\alpha} \quad as_{2\varepsilon+h2\alpha} \quad 0\rangle \quad (\text{E.28})$$

App. E.4 I_{in} with C before the polarising beam-splitter



General input vector \mathbf{I}_{in} with atmospheric backscatter and emitter and receiver optics.

$$\mathbf{I}_S = \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{C} \mathbf{M}_O \mathbf{F} \mathbf{M}_E \mathbf{I}_L \Rightarrow \mathbf{I}_{in} = \mathbf{M}_O \mathbf{F} \mathbf{I}_E \quad (\text{E.29})$$

The most complex case for the input Stokes vector \mathbf{I}_{in} is, if the calibrator is placed before the polarising beam-splitter, because here we have to multiply several matrices. All other cases can be derived from this case by neglecting the appropriate parameters (see App. D). The emitted beam Stokes vector \mathbf{I}_E from App. E.2 has to be multiplied with the atmospheric backscatter matrix \mathbf{F} (Eq. (S.3.1)) and the receiver optics matrix \mathbf{M}_O , the latter expressed as a rotated diattenuator (see Eq. (E.32)). In general the emitter optics and the laser polarisation \mathbf{I}_L are rotated as in Eq. (E.30), which is not mentioned explicitly when needless.

$$\mathbf{I}_E(\beta, \alpha) = \mathbf{M}_E(\beta) |\mathbf{I}_L(\alpha)\rangle = T_E I_L |i_E(\beta, \alpha) \quad q_E(\beta, \alpha) \quad u_E(\beta, \alpha) \quad v_E(\beta, \alpha)\rangle \quad (\text{E.30})$$

$$\begin{aligned}
\frac{\mathbf{I}_{in}(\gamma, a, \cdot)}{T_{in}I_L} &= \frac{\mathbf{M}_O(\gamma)|\mathbf{F}(a)\mathbf{M}_E\mathbf{I}_L\rangle}{T_O F_{11} T_E I_L} = \frac{\mathbf{M}_O(\gamma)|\mathbf{F}(a)\mathbf{I}_E\rangle}{T_O F_{11} T_E I_L} = |i_{in} \quad q_{in} \quad u_{in} \quad v_{in}\rangle = \\
&= \left(\begin{array}{cccc} 1 & c_{2\gamma}D_O & s_{2\gamma}D_O & 0 \\ c_{2\gamma}D_O & 1-s_{2\gamma}^2W_O & s_{2\gamma}c_{2\gamma}W_O & -s_{2\gamma}Z_Os_O \\ s_{2\gamma}D_O & s_{2\gamma}c_{2\gamma}W_O & 1-c_{2\gamma}^2W_O & c_{2\gamma}Z_Os_O \\ 0 & s_{2\gamma}Z_Os_O & -c_{2\gamma}Z_Os_O & Z_Oc_O \end{array} \right) \left| \begin{array}{c} i_E \\ aq_E \\ -au_E \\ (1-2a)v_E \end{array} \right\rangle = \\
&= \left| \begin{array}{c} i_E + D_O a (c_{2\gamma}q_E - s_{2\gamma}u_E) \\ c_{2\gamma}D_O i_E + aq_E - s_{2\gamma} \left[W_O a (s_{2\gamma}q_E + c_{2\gamma}u_E) + Z_O s_O (1-2a)v_E \right] \\ s_{2\gamma}D_O i_E - au_E + c_{2\gamma} \left[W_O a (s_{2\gamma}q_E + c_{2\gamma}u_E) + Z_O s_O (1-2a)v_E \right] \\ Z_O s_O a (s_{2\gamma}q_E + c_{2\gamma}u_E) + Z_O c_O (1-2a)v_E \end{array} \right\rangle
\end{aligned} \tag{E.31}$$

• Special cases: From Eq. (E.31) without receiver optics rotation γ we get Eq. (E.32).

$$\gamma = 0 \Rightarrow$$

$$\frac{\mathbf{I}_{in}(0, a, \cdot)}{T_{in}I_L} = \frac{\mathbf{M}_O(0)|\mathbf{F}(a)\mathbf{M}_E\mathbf{I}_L\rangle}{T_O F_{11} T_E I_L} = |i_{in} \quad q_{in} \quad u_{in} \quad v_{in}\rangle = \left| \begin{array}{c} i_E + aD_O q_E \\ D_O i_E + aq_E \\ Z_O [-c_O a u_E + s_O (1-2a)v_E] \\ Z_O [s_O a u_E + c_O (1-2a)v_E] \end{array} \right\rangle \tag{E.32}$$

• With linearly polarised laser \mathbf{I}_L with polarisation parameter a_L , with emitter optics \mathbf{M}_E , atmosphere \mathbf{F} , and receiver optics \mathbf{M}_O , and with Eqs. (E.32) and (E.20) we get Eq. (E.33).

$$i_L = q_L = 1, u_L = v_L = 0 \Rightarrow$$

$$\begin{aligned}
\frac{\mathbf{I}_{in}(\gamma, a, \beta, \alpha, a_L)}{T_{in}I_L} &= \frac{\mathbf{M}_O(\gamma)|\mathbf{F}(a)\mathbf{M}_E(\beta)\mathbf{I}_L(\alpha, a_L)\rangle}{T_O F_{11} T_E I_L} = \\
&= \left(\begin{array}{cccc} 1 & c_{2\gamma}D_O & s_{2\gamma}D_O & 0 \\ c_{2\gamma}D_O & 1-s_{2\gamma}^2W_O & s_{2\gamma}c_{2\gamma}W_O & -s_{2\gamma}Z_Os_O \\ s_{2\gamma}D_O & s_{2\gamma}c_{2\gamma}W_O & 1-c_{2\gamma}^2W_O & c_{2\gamma}Z_Os_O \\ 0 & s_{2\gamma}Z_Os_O & -c_{2\gamma}Z_Os_O & Z_Oc_O \end{array} \right) \left| \begin{array}{c} 1 + a_L D_E c_{2\alpha-2\beta} \\ a \left[c_{2\beta} D_E + a_L (c_{2\alpha} + s_{2\beta} W_E s_{2\alpha-2\beta}) \right] \\ -a \left[s_{2\beta} D_E + a_L (s_{2\alpha} - c_{2\beta} W_E s_{2\alpha-2\beta}) \right] \\ -(1-2a) a_L Z_E s_E s_{2\alpha-2\beta} \end{array} \right\rangle
\end{aligned} \tag{E.33}$$

• Eq. (E.33) with rotated, linearly polarised laser without laser depolarisation ($a_L = 1$) and rotated emitter optics (Eq. (E.20)) the input Stokes vector becomes explicitly

$$a_L = 1, i_L = q_L = 1, u_L = v_L = 0, \gamma = 0 \Rightarrow$$

$$\frac{\mathbf{I}_{in}}{T_{in}I_L} = \frac{\mathbf{M}_O(0) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha)}{T_O F_{11} T_E I_L} = \left(\begin{array}{c} (1 + D_E c_{2\alpha-2\beta}) + a D_O (c_{2\alpha} + c_{2\beta} D_E + s_{2\beta} W_E s_{2\alpha-2\beta}) \\ D_O (1 + D_E c_{2\alpha-2\beta}) + a (c_{2\alpha} + c_{2\beta} D_E + s_{2\beta} W_E s_{2\alpha-2\beta}) \\ -Z_O \left\{ s_O Z_E s_E s_{2\alpha-2\beta} + a \left[c_O (s_{2\alpha} + s_{2\beta} D_E - c_{2\beta} W_E s_{2\alpha-2\beta}) - 2s_O Z_E s_E s_{2\alpha-2\beta} \right] \right\} \\ -Z_O \left\{ c_O Z_E s_E s_{2\alpha-2\beta} - a \left[s_O (s_{2\alpha} + s_{2\beta} D_E - c_{2\beta} W_E s_{2\alpha-2\beta}) + 2c_O Z_E s_E s_{2\alpha-2\beta} \right] \right\} \end{array} \right) \quad (E.34)$$

• Eq. (E.34) with laser polarisation and emitter optics aligned

$$a_L = 1, i_L = q_L = 1, u_L = v_L = 0, \gamma = 0, \beta = \alpha \Rightarrow$$

$$\frac{\mathbf{I}_{in}}{T_{in}I_L} = \frac{\mathbf{M}_O(0) \mathbf{F}(a) \mathbf{M}_E(\alpha) \mathbf{I}_L(\alpha)}{T_O F_{11} T_E I_L} = (1 + D_E) \left(\begin{array}{c} 1 + a D_O c_{2\alpha} \\ D_O + a c_{2\alpha} \\ -Z_O a c_O s_{2\alpha} \\ +Z_O a s_O s_{2\alpha} \end{array} \right) \quad (E.35)$$

• and without any optics and laser rotation

$$a_L = 1, i_L = q_L = 1, u_L = v_L = 0, \alpha = \beta = \gamma = 0 \Rightarrow$$

$$\frac{\mathbf{I}_{in}(0,0,0,0,1)}{T_{in}I_L} = \frac{\mathbf{M}_O(0) \mathbf{F}(a) \mathbf{M}_E(0) \mathbf{I}_L(0)}{T_O F_{11} T_E I_L} = (1 + D_E) |1 + a D_O \quad D_O + a \quad 0 \quad 0\rangle \quad (E.36)$$

• Eq. (E.33) without emitter optics \mathbf{M}_E

$$D_E = 0, s_E = 0, W_E = 0, a' = a a_L \Rightarrow$$

$$\frac{\mathbf{I}_{in}(\gamma, a, 0, \alpha, a_L)}{T_{in}I_L} = \frac{\mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{I}_L(\alpha, a_L)}{T_O F_{11} I_L} = \left(\begin{array}{c} 1 + c_{2\gamma+2\alpha} a' D_O \\ c_{2\gamma} D_O + a' [c_{2\alpha} - s_{2\gamma} s_{2\gamma+2\alpha} W_O] \\ s_{2\gamma} D_O - a' [s_{2\alpha} - c_{2\gamma} s_{2\gamma+2\alpha} W_O] \\ s_{2\gamma+2\alpha} a' Z_O s_O \end{array} \right) \quad (E.37)$$

• No emitter optics \mathbf{M}_E and no receiver optics rotation

$$\text{with } \gamma = 0, T_E = 1, D_E = 0, s_E = 0, W_E = 0, a' = a a_L \Rightarrow$$

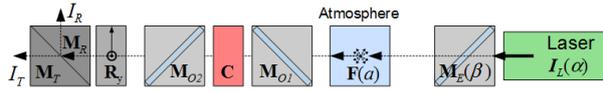
$$\frac{\mathbf{I}_{in}(0, a, 0, \alpha, a_L)}{T_{in}I_L} = \frac{\mathbf{M}_O(0) \mathbf{F}(a) \mathbf{I}_L(\alpha, a_L)}{T_O F_{11} I_L} = |1 + c_{2\alpha} a' D_O \quad D_O + c_{2\alpha} a' \quad -s_{2\alpha} a' Z_O c_O \quad s_{2\alpha} a' Z_O s_O\rangle \quad (E.38)$$

• The latter and no laser rotation

$$\text{with } \alpha = 0, \gamma = 0, T_E = 1, D_E = 0, s_E = 0, W_E = 0, a' = a a_L \Rightarrow$$

$$\frac{\mathbf{I}_{in}(0, a, 0, 0, a_L)}{T_{in}I_L} = \frac{\mathbf{M}_O(0) \mathbf{F}(a) \mathbf{I}_L(0, a_L)}{T_O F_{11} I_L} = |1 + a' D_O \quad D_O + a' \quad 0 \quad 0\rangle \quad (E.39)$$

App. E.5 I_{in} with C amidst the receiving optics



In case there is polarising or/and retarding optics before (\mathbf{M}_{O1}) and after (\mathbf{M}_{O2}) the calibrator as in Eq. (E.40), the basic equations can be constructed by using the analyser matrix \mathbf{A}_S from App. D.2 and the input Stokes vectors \mathbf{I}_{in} from App. E.4.

$$\mathbf{I}_S = \eta_S \mathbf{M}_S \mathbf{R}_y \mathbf{M}_{O2} \mathbf{C} \mathbf{M}_{O1} \mathbf{F} \mathbf{I}_E \Rightarrow \mathbf{A}_S = \mathbf{M}_S \mathbf{R}_y \mathbf{M}_{O2} \quad \text{and} \quad \mathbf{I}_{in} = \mathbf{M}_{O1} \mathbf{F} \mathbf{I}_E \quad (\text{E.40})$$

Supplement to amt-2015-338

S.1 Coordinate system and conventions

Müller matrices describe the effect of optical elements on the Stokes vector with respect to a coordinate system and using a set of definitions about signs and directions. Different sets of definitions can be found in the literature as discussed in detail in Muller (1969); some of which are even inconsistent. The discussions led to the so-called *Muller (or Muller-Nebraska-) convention*, which we follow in this paper (see also Hauge et al. (1980)). We use a right-handed Cartesian coordinate system (see Fig. 8), in which angles are defined counter-clock wise, i.e. from the x- to the y-axis, when looking against the z-axis. The local z-axis points in the propagation direction of the light. We define the reference coordinate system of the lidar setup by the orientation of the polarising beam-splitter (PBS) in the receiving optics. Light polarised with its E-vector on the x-axis, i.e. parallel to the incident plane of the PBS in Fig. 7, is mostly transmitted by a usual PBS, while light with polarisation in y-direction, i.e. perpendicularly polarised to the incident plane, is mostly reflected. The incident plane is spanned up by the direction of light propagation (z-axis, propagation vector \mathbf{k}) and the normal of the reflecting surface, which means that the incident plane in Fig. 7 is the x-z-plane). The parallel and perpendicular polarisations are also called the p- and s-polarisation, respectively. The orientation of linearly polarised light is defined by the orientation of the plane of vibration, which contains both the electric vector \mathbf{E} and the propagation vector \mathbf{k} .

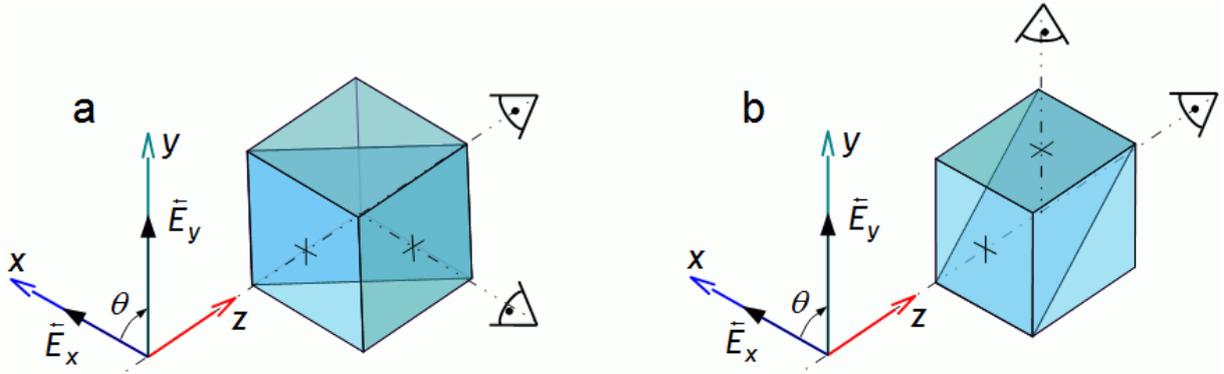


Fig. 7 Definition of the reference coordinate system with respect to the incidence plane of the polarising beam-splitter.

Other Müller matrix measurement configurations may have other arrangements for the coordinates. All choices, however, are arbitrary, and lead to different Müller matrices (Chipman 2009b). There is no preferred set of definitions in the literature. According to our choice of orientation, the diattenuation parameter D is defined as

$$D_T = \frac{T_T^p - T_T^s}{T_T^p + T_T^s}, \quad \Delta_T = \Delta_T^p - \Delta_T^s,$$

$$D_R = \frac{T_R^p - T_R^s}{T_R^p + T_R^s}, \quad \Delta_R = \Delta_R^p - \Delta_R^s$$

In order to keep the results of the Müller matrix calculations consistent when adding reflecting surfaces as mirrors and beam-splitters in the optical setup, a right-handed xyz-

coordinate system is used with the z-axis in the direction of the light propagation. The vertical (perpendicular) polarised light has its E-vector in y-direction,

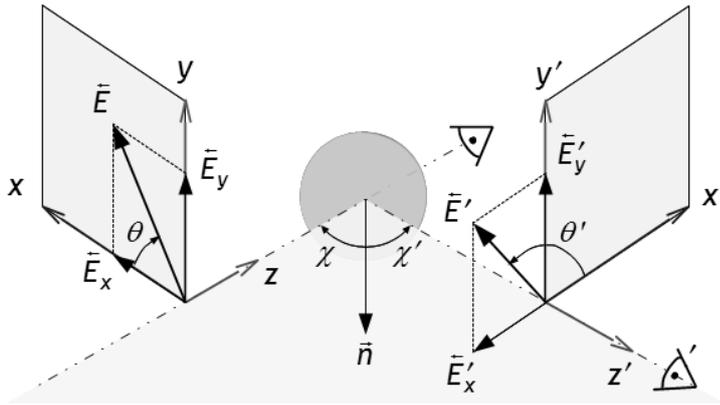


Fig. 8 Reflection of a Stokes vector.

S.2 Stokes vector and Müller matrix

The Stokes vector and the Müller matrix are one representation of the state of polarisation of light, which is based on measurable quantities. The Stokes vector describes the polarisation state of a light beam, and the Müller matrix describes how the Stokes vector changes when passing through an optical volume, which can be an optical element or an atmospheric path with scattering, absorbing and refracting properties. A Stokes vector can be determined by six measurements of the flux I with ideal linear and circular polarisation analysers at different orientations before a detector (Chipman 2009a; Ch. 15.17):

- I^p parallel (horizontal) linear polarizer (0°)
- I^s perpendicular (vertical) linear polarizer (90°)
- I^{45} 45° linear polarizer
- I^{135} 135° linear polarizer
- I^R right circular polarizer
- I^L left circular polarizer

(S.2.1)

The Stokes vector is defined as

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I^p + I^s \\ I^p - I^s \\ I^{45} - I^{135} \\ I^R - I^L \end{pmatrix} \quad (\text{S.2.2})$$

Right-circularly polarised light is defined as a clockwise rotation of the electric vector when the observer is looking against the direction of light propagation (Bennett 2009a). Another representation, the so-called modified Stokes column vector (Mishchenko et al. 2002), uses the horizontally (parallel, p) and vertically (perpendicular, s) polarised fluxes I_p and I_s as the first two Stokes parameters. They can be transformed from the Stokes vector with a transformation matrix

$$\begin{pmatrix} I_p \\ I_s \\ U \\ V \end{pmatrix} = \mathbf{A} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} \Rightarrow \begin{aligned} I_p &= 0.5(I + Q) \\ I_s &= 0.5(I - Q) \end{aligned} \quad (\text{S.2.3})$$

and vice versa

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} I_p \\ I_s \\ U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_p \\ I_s \\ U \\ V \end{pmatrix} \quad (\text{S.2.4})$$

For fully polarised light, the Stokes parameters fulfil the equation

$$I^2 = Q^2 + U^2 + V^2 \quad (\text{S.2.5})$$

and for full linearly polarised light

$$I^2 = Q^2 + U^2 \quad (\text{S.2.6})$$

S.3 Depolarisation

Depolarisation is closely related to scattering and usually has its origins from retardance or diattenuation which is rapidly varying in time, wavelength, or spatially over an optical device (Cornu-, Lyot-, or wedge-depolariser). Depolarisation causes a loss of coherence of the polarisation state (Chipman 2009a). The polarisation vector \mathbf{I}_F reflected by the atmosphere $\mathbf{F}(a)$ with linear polarisation parameter a from a generally polarised laser \mathbf{I}_L is (van de Hulst 1981; Sect. 5.32) (Mishchenko et al. 2002; Sect. 4).

$$\frac{\mathbf{I}_F(a)}{F_{11}T_E I_L} = \frac{\mathbf{F}(a)|\mathbf{M}_E \mathbf{I}_L\rangle}{F_{11}T_E I_L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1-2a \end{pmatrix} \begin{pmatrix} i_E \\ q_E \\ u_E \\ v_E \end{pmatrix} = \begin{pmatrix} i_E \\ aq_E \\ -au_E \\ (1-2a)v_E \end{pmatrix} \quad (\text{S.3.1})$$

The linear depolarisation ratio is defined as

$$\delta = \frac{F_{11} - F_{22}}{F_{11} + F_{22}} = \frac{1-a}{1+a} \Rightarrow a = \frac{1-\delta}{1+\delta} \quad (\text{S.3.2})$$

With a linearly polarised laser with intensity I_L and linear polarisation parameter a_L and rotational misalignment α , i.e. without emitter optics, the laser light reflected by the atmosphere with linear polarisation parameter a is

$$\frac{\mathbf{I}_F(a, \alpha, a_L)}{F_{11}I_L} = \frac{\mathbf{F}(a)|\mathbf{I}_L(\alpha, a_L)\rangle}{F_{11}I_L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1-2a \end{pmatrix} \begin{pmatrix} 1 \\ a_L c_{2\alpha} \\ a_L s_{2\alpha} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ aa_L c_{2\alpha} \\ -aa_L s_{2\alpha} \\ 0 \end{pmatrix} \quad (\text{S.3.3})$$

It is obvious that the lasers a_L and the atmospheres a cannot be discerned in the resulting Stokes vector, and the measured, combined polarisation parameter is

$$a' = aa_L \quad (\text{S.3.4})$$

The linear depolarisation ratio δ' resulting from a' can be retrieved with Eq. (12)

$$\delta' = \frac{1 - a'}{1 + a'} = \frac{\delta + \delta_L}{1 + \delta\delta_L} \quad (\text{S.3.5})$$

For a small linear depolarisation ratio δ_L of the laser beam, the resulting linear depolarisation ratio of an atmospheric measurement is about the sum of the lasers and the atmospheres linear depolarisation ratios

$$\delta_L \ll 1 \Rightarrow \delta' \approx \delta + \delta_L \quad (\text{S.3.6})$$

If δ_L is unknown, the uncertainty will cause an absolute error of the finally retrieved atmospheric linear depolarisation ratio.

S.4 Retarding linear diattenuator

The diattenuation magnitude D^* of an optical element, usually simply *diattenuation*, is calculated from the maximum and minimum transmitted intensities I (or transmittances T) (Chipman 2009b), measured by rotating a linearly polarising analyser in front of the element:

$$D^* = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{T_{\max} - T_{\min}}{T_{\max} + T_{\min}} \quad (\text{S.4.1})$$

The diattenuation magnitude D^* is always positive, and if D^* is deduced from the reflectances T_R^p and T_R^s of an optical sample as in Eq. (17), Eq. (S.4.1) becomes (Lu and Chipman 1996)

$$D_R^* \equiv \left| \frac{T_R^p - T_R^s}{T_R^p + T_R^s} \right| \quad (\text{S.4.2})$$

In order to avoid sign changes in the equations between the cases where $T_R^p < T_R^s$ and $T_R^p > T_R^s$, we use instead the diattenuation parameter D_S (Eq. (S.4.3); see Chipman (2009b)), where it is named d_x or d_h), with which all equations can be expressed together for the transmitting (subscript $S = T$) and the reflecting (subscript $S = R$) part of a polarising beam-splitter.

$$D_S \in \{D_T, D_R\}, \quad D_R \equiv \frac{T_R^p - T_R^s}{T_R^p + T_R^s}, \quad D_T \equiv \frac{T_T^p - T_T^s}{T_T^p + T_T^s} \quad (\text{S.4.3})$$

The transmittances for unpolarised light are shown in Eq. (S.4.4), and some often occurring expressions in Eqs. (S.4.5) and (S.4.6).

$$T_R \equiv \frac{T_R^p + T_R^s}{2}, \quad T_T \equiv \frac{T_T^p + T_T^s}{2} \quad (\text{S.4.4})$$

$$1 - D_R = T_R^s / T_R, \quad 1 + D_R = T_R^p / T_R, \quad 1 - D_T = T_T^s / T_T, \quad 1 + D_T = T_T^p / T_T \quad (\text{S.4.5})$$

$$\frac{T_R^s}{T_R^p} = \frac{1 - D_R}{1 + D_R}, \quad \frac{T_T^s}{T_T^p} = \frac{1 - D_T}{1 + D_T} \quad (\text{S.4.6})$$

The optical elements considered here are non-depolarising, linear diattenuators \mathbf{M}_D , with linear diattenuation parameter D_O and average transmission T_O for unpolarised light, combined with linear retarders \mathbf{M}_{Ret} (linear retardance ΔO , $\cos \Delta O = c_O$, $\sin \Delta O = s_O$). The optical elements with possibly considerable diattenuation and retardation are dichroic beam-splitters, which are used to separate the wavelengths and to analyse the state of polarisation of the

collimated beam in the receiver optics. They are used in transmission and reflection. The matrix of the transmitting part is Eq. (S.4.7) (see Eqs. (14, 15))

$$\mathbf{M}_T = \frac{1}{2} \begin{pmatrix} T_T^p + T_T^s & T_T^p - T_T^s & 0 & 0 \\ T_T^p - T_T^s & T_T^p + T_T^s & 0 & 0 \\ 0 & 0 & 2\sqrt{T_T^p T_T^s} \cos \Delta_T & 2\sqrt{T_T^p T_T^s} \sin \Delta_T \\ 0 & 0 & -2\sqrt{T_T^p T_T^s} \sin \Delta_T & 2\sqrt{T_T^p T_T^s} \cos \Delta_T \end{pmatrix} =$$

$$= T_T \begin{pmatrix} 1 & D_T & 0 & 0 \\ D_T & 1 & 0 & 0 \\ 0 & 0 & Z_T c_T & Z_T s_T \\ 0 & 0 & -Z_T s_T & Z_T c_T \end{pmatrix} \quad (\text{S.4.7})$$

$$Z_T \equiv \frac{2\sqrt{T_T^p T_T^s}}{T_T^p + T_T^s} = \sqrt{1 - D_T^2}, \quad c_T \equiv \cos \Delta_T, \quad s_T \equiv \sin \Delta_T, \quad \Delta_T \equiv \varphi_T^p - \varphi_T^s \quad (\text{S.4.8})$$

with the short-cuts in Eq. (S.4.8), the intensity transmission coefficients (transmittance) for light polarised parallel (T_T^p) and perpendicular (T_T^s) to the plane of incidence of the PBS, the diattenuation parameter D_T (see Suppl. S.3), and the average transmittance T_T for unpolarised light. Δ_T is the difference of the phase shifts of the parallel and perpendicular polarised electrical fields. The Müller matrix for the reflecting part of the PBS (see Eqs. 16,17) includes a mirror reflection (Suppl. S.6):

$$\mathbf{M}_R = T_R \begin{pmatrix} 1 & D_R & 0 & 0 \\ D_R & 1 & 0 & 0 \\ 0 & 0 & -Z_R c_R & -Z_R s_R \\ 0 & 0 & Z_R s_R & -Z_R c_R \end{pmatrix} = T_R \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & D_R & 0 & 0 \\ D_R & 1 & 0 & 0 \\ 0 & 0 & Z_R c_R & Z_R s_R \\ 0 & 0 & -Z_R s_R & Z_R c_R \end{pmatrix} \quad (\text{S.4.9})$$

with the corresponding intensity reflection coefficients (reflectance) for light polarised parallel (T_R^p) and perpendicular (T_R^s) to the plane of incidence of the PBS and

$$Z_R \equiv \frac{2\sqrt{T_R^p T_R^s}}{T_R^p + T_R^s} = \sqrt{1 - D_R^2}, \quad c_R \equiv \cos \Delta_R, \quad s_R \equiv \sin \Delta_R, \quad \Delta_R \equiv \varphi_R^p - \varphi_R^s \quad (\text{S.4.10})$$

In order to simplify the derivation of the equations, we write in the following for both, the reflecting and transmitting matrix of the polarising beam-splitter, \mathbf{M}_S (subscript S for splitter) where appropriate.

$$Z_S \in \{-Z_R, Z_T\}, \quad \mathbf{M}_S \in \{\mathbf{M}_R, \mathbf{M}_T\}, \quad I_S \in \{I_R, I_T\} \quad (\text{S.4.11})$$

Müller matrices as in Eqs. (S.4.7) and (S.4.9) can be decomposed in matrices of a pure diattenuator \mathbf{M}_D and a pure retarder \mathbf{M}_{ret} :

$$\mathbf{M}_O = \mathbf{M}_D \mathbf{M}_{ret} = T_O \begin{pmatrix} 1 & D_O & 0 & 0 \\ D_O & 1 & 0 & 0 \\ 0 & 0 & Z_O & 0 \\ 0 & 0 & 0 & Z_O \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_O & s_O \\ 0 & 0 & -s_O & c_O \end{pmatrix} = T_O \begin{pmatrix} 1 & D_O & 0 & 0 \\ D_O & 1 & 0 & 0 \\ 0 & 0 & Z_O c_O & Z_O s_O \\ 0 & 0 & -Z_O s_O & Z_O c_O \end{pmatrix} \quad (\text{S.4.12})$$

As both are linear, they commute (Eq. (S.4.13)). They have a block-diagonal structure.

$$\mathbf{M}_O = \mathbf{M}_D \mathbf{M}_{ret} = \mathbf{M}_{ret} \mathbf{M}_D \quad (\text{S.4.13})$$

Among such optical elements are $\lambda/4$ plates (Suppl. S.10.16), $\lambda/2$ plates (Suppl. S.10.13), dichroic beam-splitters, polarising beam-splitters, polarising sheet filters, aluminium and dielectric mirrors, and also uncoated glass surfaces under oblique incident angles. For further information see Azzam (2009); Bennett (2009a,b); Chipman (2009b,a)

S.5 Rotation

S.5.1 Rotation about the direction of light propagation

Some confusion can arise because rotation about the optical axis can be done on a Stokes vector, on the coordinate system (coordinate transformation), and on an optical element while keeping the reference coordinate system. The first two rotations don't change the state of the circular polarisation, while a rotated optical element can do that. Additional confusion arises because often in the literature and in textbooks the vector and coordinate rotations are mixed, or the derivation of the presented final equations from first principles are not provided, and sometimes the explanatory text is misleading or inconsistent. We follow the notations in Mishchenko et al. (2002); Chipman (2009b). Rotations are anti-clockwise, from the x -axis towards the y -axis, seen against the direction of light propagation (z -axis).

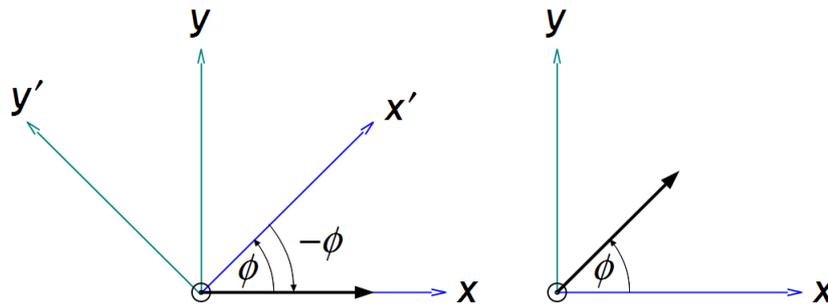


Fig. 9 Rotation of the xy -coordinate system (left) and of a vector (right). The z -axis, i.e. the direction of light propagation, points out of the paper plane.

A Stokes vector which is physically rotated by an angle ϕ while the coordinate system is fixed becomes (Mishchenko et al. 2002; Ch. 1.5)

$$\mathbf{I}(\phi) = \mathbf{R}(\phi) \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -s_{2\phi} & 0 \\ 0 & s_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I \\ c_{2\phi} Q - s_{2\phi} U \\ s_{2\phi} Q + c_{2\phi} U \\ V \end{pmatrix} \quad (\text{S.5.1.1})$$

with the abbreviations

$$c_{2\phi} \equiv \cos 2\phi, \quad s_{2\phi} \equiv \sin 2\phi, \quad (\text{S.5.1.2})$$

and the rotation matrix $\mathbf{R}(\phi)$

$$\mathbf{R}(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -s_{2\phi} & 0 \\ 0 & s_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{S.5.1.3})$$

$$\mathbf{R}(90^\circ + \phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -c_{2\phi} & s_{2\phi} & 0 \\ 0 & -s_{2\phi} & -c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{S.5.1.4})$$

$$\mathbf{R}(\pm 45^\circ + \varepsilon) = \mathbf{R}(x45^\circ + \varepsilon) = \mathbf{R}(x, \varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -xs_{2\varepsilon} & -xc_{2\varepsilon} & 0 \\ 0 & xc_{2\varepsilon} & -xs_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{S.5.1.5})$$

With these definitions a formula for one rotation can easily be converted to other angles with

$$\begin{aligned} \Psi &\rightarrow 0^\circ + \varepsilon &\leftrightarrow &\pm 45^\circ + \varepsilon \\ \mathbf{R}(\Psi) &\rightarrow \mathbf{R}(\varepsilon) &\leftrightarrow &\mathbf{R}(x45^\circ + \varepsilon) \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\Psi} & -s_{2\Psi} & 0 \\ 0 & s_{2\Psi} & c_{2\Psi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\varepsilon} & -s_{2\varepsilon} & 0 \\ 0 & s_{2\varepsilon} & c_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} &\leftrightarrow &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -xs_{2\varepsilon} & -xc_{2\varepsilon} & 0 \\ 0 & xc_{2\varepsilon} & -xs_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ c_{2\Psi} &\rightarrow c_{2\varepsilon} &\leftrightarrow &-xs_{2\varepsilon} \\ s_{2\Psi} &\rightarrow s_{2\varepsilon} &\leftrightarrow &xc_{2\varepsilon} \end{aligned} \quad (\text{S.5.1.6})$$

Please note, that in Mishchenko et al. (2002; Ch. 1.5) the equations describe a rotation of the Stokes vector, while the text specifies the transformation as "rotation of the two-dimensional coordinate system". The two transformations are called "alibi" and "alias" transformation (Steinborn and Ruedenberg 1973), respectively. The Stokes vector rotates contra-variantly under the change of the basis. If we rotate the coordinate system (alias transformation) by an angle ϕ (see Fig. 9), the original Stokes vector \mathbf{I} appears in the rotated coordinate system under the angle $-\phi$, and the Stokes vector \mathbf{I}' in the rotated coordinate system is Eq. (S.5.1.7).

$$\mathbf{I}' = \begin{pmatrix} I' \\ Q' \\ U' \\ V' \end{pmatrix} = \mathbf{R}(-\phi) \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & s_{2\phi} & 0 \\ 0 & -s_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I \\ c_{2\phi} Q + s_{2\phi} U \\ -s_{2\phi} Q + c_{2\phi} U \\ V \end{pmatrix} \quad (\text{S.5.1.7})$$

The rotation of the polarisation of a Stokes vector can be accomplished by means of a $\lambda/2$ plate (HWP), which is a 180° linear retarder. An ideal HWP can be derived from Eq. (S.5.2.3) by setting $\Delta_\theta = 180^\circ$, $D_\theta = 0 \Rightarrow Z_\theta = 1$ and $W_\theta = 2$, and $T_\theta = 1$:

$$\begin{aligned}
\mathbf{M}_{HWP}(\phi) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2s_{2\phi}^2 & 2s_{2\phi}c_{2\phi} & 0 \\ 0 & 2s_{2\phi}c_{2\phi} & 1-2c_{2\phi}^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{4\phi} & s_{4\phi} & 0 \\ 0 & s_{4\phi} & -c_{4\phi} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \\
&= \mathbf{R}(2\phi) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned} \tag{S.5.1.8}$$

A HWP rotates a Stokes vector by twice the own rotation and additionally changes the direction of the circularly polarised component. For a rotation of $\phi = 90^\circ$ the HWP acts as a mirror but without changing the direction of light propagation. Real HWPs are often made of birefringent crystals. Their retardance depends in general on the wavelength, on the incident angle, and on the temperature. For lidar applications so-called true zero-order HWPs are best suited because of their relative insensitivity to temperature and incidence angle. The matrices for the HWP rotator and for the mechanical rotator can be combined in one matrix \mathbf{M}_{rot} as shown in Suppl. S.10.15.

S.5.2 Rotation of a retarding diattenuator

The rotation of an optical element with Müller matrix \mathbf{M}_o by an angle ϕ about the direction of light propagation is mathematically performed by first rotating the coordinate system before \mathbf{M}_o by $-\phi$, to achieve the description of the Stokes vector in the local coordinate system (eigen-polarisations) of \mathbf{M}_o , and then rotating the coordinate system behind \mathbf{M}_o back to the reference coordinate system by ϕ using the rotation matrix $\mathbf{R}(\phi)$

$$\mathbf{M}_o(\phi) = \mathbf{R}(\phi) \mathbf{M}_o(0^\circ) \mathbf{R}(-\phi), \tag{S.5.2.1}$$

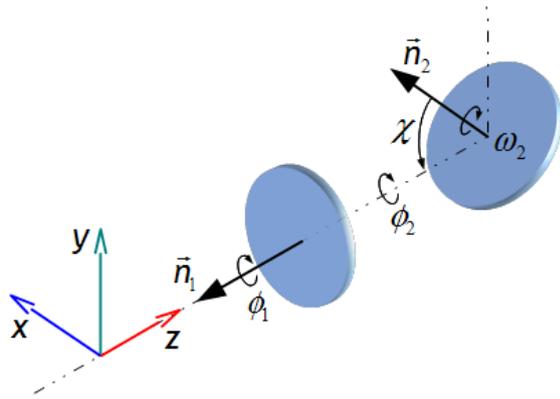


Figure 10 Rotation angles of an optical element. The rotations considered in this work are only ϕ_1 and ϕ_2 .

A linear retarding diattenuator \mathbf{M}_o rotated by ϕ about the z-axis becomes

$$\begin{aligned}
\mathbf{M}_O(\phi) &= \mathbf{R}(\phi)\mathbf{M}_O\mathbf{R}(-\phi) = \\
&= T_O \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -s_{2\phi} & 0 \\ 0 & s_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & D_O & 0 & 0 \\ D_O & 1 & 0 & 0 \\ 0 & 0 & Z_O c_O & Z_O s_O \\ 0 & 0 & -Z_O s_O & Z_O c_O \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & s_{2\phi} & 0 \\ 0 & -s_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\
&= T_O \begin{pmatrix} 1 & c_{2\phi} D_O & s_{2\phi} D_O & 0 \\ c_{2\phi} D_O & 1 - s_{2\phi}^2 W_O & s_{2\phi} c_{2\phi} W_O & -s_{2\phi} Z_O s_O \\ s_{2\phi} D_O & s_{2\phi} c_{2\phi} W_O & 1 - c_{2\phi}^2 W_O & c_{2\phi} Z_O s_O \\ 0 & s_{2\phi} Z_O s_O & -c_{2\phi} Z_O s_O & Z_O c_O \end{pmatrix}
\end{aligned} \tag{S.5.2.2}$$

$$c_{2\phi} = \cos 2\phi, s_{2\phi} = \sin 2\phi, c_O = \cos \Delta_O, s_O = \sin \Delta_O, Z_O \equiv \sqrt{1 - D_O^2}, W_O = 1 - Z_O c_O \tag{S.5.2.3}$$

Without diattenuation we get

$$D_O = 0 \Rightarrow Z_O = \sqrt{1 - D_O^2} = 1, W_O = 1 - c_O \tag{S.5.2.4}$$

and with ideal diattenuation

$$|D_O| = 1, Z_O = \sqrt{1 - D_O^2} = 0, W_O = 1 \tag{S.5.2.5}$$

Rotation of a retarding diattenuator by $\pm 45^\circ + \varepsilon$

$$\begin{aligned}
\mathbf{M}_O(x45^\circ + \varepsilon) &= \mathbf{R}(x45^\circ)\mathbf{M}_O(\varepsilon)\mathbf{R}(-x45^\circ) = \\
&= T_O \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & -x & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_{2\varepsilon} D_O & s_{2\varepsilon} D_O & 0 \\ c_{2\varepsilon} D_O & 1 - s_{2\varepsilon}^2 W_O & s_{2\varepsilon} c_{2\varepsilon} W_O & -s_{2\varepsilon} Z_O s_O \\ s_{2\varepsilon} D_O & s_{2\varepsilon} c_{2\varepsilon} W_O & 1 - c_{2\varepsilon}^2 W_O & c_{2\varepsilon} Z_O s_O \\ 0 & s_{2\varepsilon} Z_O s_O & -c_{2\varepsilon} Z_O s_O & Z_O c_O \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\
&= T_O \begin{pmatrix} 1 & x s_{2\varepsilon} D_O & -x c_{2\varepsilon} D_O & 0 \\ x s_{2\varepsilon} D_O & 1 - c_{2\varepsilon}^2 W_O & -s_{2\varepsilon} c_{2\varepsilon} W_O & x c_{2\varepsilon} Z_O s_O \\ -x c_{2\varepsilon} D_O & -s_{2\varepsilon} c_{2\varepsilon} W_O & 1 - s_{2\varepsilon}^2 W_O & x s_{2\varepsilon} Z_O s_O \\ 0 & -x c_{2\varepsilon} Z_O s_O & -x s_{2\varepsilon} Z_O s_O & Z_O c_O \end{pmatrix} = \mathbf{R}(\varepsilon)\mathbf{M}_O(x45^\circ)\mathbf{R}(-\varepsilon)
\end{aligned} \tag{S.5.2.6}$$

and without error angle ε :

$$\mathbf{M}_O(x45^\circ) = \mathbf{R}(x45^\circ)\mathbf{M}_O\mathbf{R}(-x45^\circ) = X_O \begin{pmatrix} 1 & 0 & x D_O & 0 \\ 0 & Z_O c_O & 0 & -x Z_O s_O \\ x D_O & 0 & 1 & 0 \\ 0 & x Z_O s_O & 0 & Z_O c_O \end{pmatrix} \tag{S.5.2.7}$$

S.6 Mirror

For a pure mirror without diattenuation or retardance the Müller matrix is

$$\mathbf{M}_M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{S.6.1})$$

which results from the rotation of the detector (symbolised by the eye) about the y-axis from the rear of the optical element to the front as shown in Fig. 11.

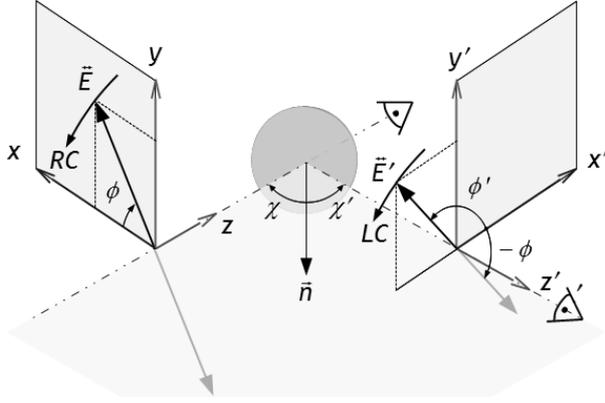


Figure 11: Reflection of light by a mirror. The light propagation is along the z-axis. The plane of vibration of linearly polarised light is indicated by the E-vectors, and right and left circular polarised light by the RC and LC arrows, respectively.

To explain the change of the axes, let the plane of vibration of linearly polarised light be rotated in the (xyz) coordinate system by ϕ around the z-axis, indicated by the E -vector in Fig. 11, and the incident angle be $\chi = 0$ for reflection from a mirror. After the mirror the direction of light propagation has changed, but not the orientation of the plane of vibration, indicated by the E'-vector. Hence, the rotation ϕ' in the mirrored coordinate system $(xyz)'$ is $\phi' = 180^\circ - \phi$, which is equivalent to $\phi' = -\phi$. Thus a Stokes vector rotated by $\mathbf{R}(\phi)$ in (xyz) is described in $(xyz)'$ after the mirror \mathbf{M}_M by

$$\mathbf{I}' = \mathbf{M}_M \mathbf{R}(\phi) \mathbf{I} = \mathbf{R}(-\phi) \mathbf{M}_M \mathbf{I} \quad (\text{S.6.2})$$

Furthermore, the circular polarisation has changed its sign from right circular (RC) before to left circular (LC) after the mirror.

S.6.1 Real mirror

Real mirrors are dielectric or metal surfaces which can exhibit considerable phase retardation and diattenuation under oblique incident angles. Hence, a real mirror is a linear retarding diattenuator \mathbf{M}_O combined with a mirror \mathbf{M}_M .

$$\begin{aligned} \mathbf{M}_{MO} &= \\ &= T_O \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & D_o & 0 & 0 \\ D_o & 1 & 0 & 0 \\ 0 & 0 & Z_o c_o & Z_o s_o \\ 0 & 0 & -Z_o s_o & Z_o c_o \end{pmatrix} = T_O \begin{pmatrix} 1 & D_o & 0 & 0 \\ D_o & 1 & 0 & 0 \\ 0 & 0 & -Z_o c_o & -Z_o s_o \\ 0 & 0 & Z_o s_o & -Z_o c_o \end{pmatrix} \end{aligned} \quad (\text{S.6.1.1})$$

which commute

$$\mathbf{M}_{MO} = \mathbf{M}_M \mathbf{M}_O = \mathbf{M}_O \mathbf{M}_M = \mathbf{M}_{OM} \quad (\text{S.6.1.2})$$

Eq. (S.6.1.1) is also the description of the reflecting part of a polarising beam-splitter or of any dichroic beam-splitter.

S.6.2 Rotation of a reflecting surface

If we rotate \mathbf{M}_{MO} , we have to mind the change of the coordinate system after the mirror. Here it is important which element comes first, because, as explained above, applying a mirror means a change of the local coordinate system after the mirror, and rotation of elements are always done with respect to the local coordinate system before the element. Hence, a diattenuator rotated in (xyz) plus a mirror described in $(xyz)'$ is using (S.5.1.5) and (S.6.2)

$$\mathbf{M}_M \mathbf{M}_O(\phi) = \mathbf{M}_M \mathbf{R}(\phi) \mathbf{M}_O \mathbf{R}(-\phi) = \mathbf{R}(-\phi) \mathbf{M}_M \mathbf{M}_O \mathbf{R}(-\phi) = \mathbf{R}(-\phi) \mathbf{M}_{MO} \mathbf{R}(-\phi) = \mathbf{M}_{MO}(\phi) \quad (\text{S.6.2.1})$$

Moving the mirror before the diattenuator

$$\mathbf{M}_O(\phi) \mathbf{M}_M = \mathbf{R}(\phi) \mathbf{M}_O \mathbf{R}(-\phi) \mathbf{M}_M = \mathbf{R}(\phi) \mathbf{M}_O \mathbf{M}_M \mathbf{R}(\phi) = \mathbf{R}(\phi) \mathbf{M}_{OM} \mathbf{R}(\phi) = \mathbf{M}_{MO}(-\phi) \quad (\text{S.6.2.2})$$

we see from Eqs. (S.6.1.2) to (S.6.2.2) that

$$\mathbf{M}_O(\phi) \mathbf{M}_M = \mathbf{M}_{OM}(\phi) = \mathbf{M}_M \mathbf{M}_O(-\phi) = \mathbf{M}_{MO}(-\phi). \quad (\text{S.6.2.3})$$

This explains why the rotation of a reflecting diattenuator has to be described as shown by Chipman (2009b), i.e.:

$$\mathbf{M}_{OM}(\phi) = \mathbf{R}(\phi) \mathbf{M}_{OM} \mathbf{R}(\phi). \quad (\text{S.6.2.4})$$

S.6.3 Beam-splitters and mirrors in the optical path

In order to make the equations developed in this work applicable to a variety of lidar systems, we have to investigate how the equations change when individual elements are changed from transmitting to reflecting. This is also useful when the reflected and the transmitted paths after a beam-splitter are to be described with the same equations.

Above we showed the local coordinate change behind a mirror. But how does this effect the outcome of a lidar measurement and of the calibration measurements? Let's consider a chain of rotated optical elements using the eigen-polarisations of the polarising beam-splitter matrix \mathbf{M}_S as the reference coordinate system

$$\mathbf{I}_S = \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma) \mathbf{M}_2(\phi) \mathbf{M}_1(\varepsilon) \mathbf{F} \mathbf{I}_{in} \quad (\text{S.6.3.1})$$

When we exchange \mathbf{M}_2 with its reflecting counterpart $\mathbf{M}_M \mathbf{M}_2$, we can move the ideal mirror \mathbf{M}_M step by step to the right in the chain using (S.6.2.3)

$$\begin{aligned} \mathbf{I}'_S &= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma) \mathbf{M}_M \mathbf{M}_2(\phi) \quad \mathbf{M}_1(\varepsilon) \quad \mathbf{F} \mathbf{I}_L(\alpha) = \\ &= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma) \quad \mathbf{M}_2(-\phi) \mathbf{M}_M \quad \mathbf{M}_1(\varepsilon) \quad \mathbf{F} \mathbf{I}_L(\alpha) = \\ &= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma) \quad \mathbf{M}_2(-\phi) \quad \mathbf{M}_1(-\varepsilon) \mathbf{M}_M \mathbf{F} \mathbf{I}_L(\alpha) = \\ &= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma) \quad \mathbf{M}_2(-\phi) \quad \mathbf{M}_1(-\varepsilon) \quad \mathbf{F} \mathbf{I}_L^*(-\alpha) \end{aligned} \quad (\text{S.6.3.2})$$

and see that all rotation angles before the changed element are inverted. In the last step of Eq. (S.6.3.2) the depolarising atmospheric \mathbf{F} -matrix is rotational invariant, and the circular polarisation of the input Stokes vector changes its sign, indicated by the star.

In other words: equations, which are derived for the system in Eq. (S.6.3.1), can be used for the system with an additional mirror as in Eq. (S.6.3.2) by inverting in the original equations all rotation angles before the mirror and reversing the circular polarisation of the input Stokes vector. In case two surfaces are changed from transmitting to reflecting as

$$\begin{aligned}
\mathbf{I}_S'' &= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma) \mathbf{M}_M \mathbf{M}_2(\phi) \mathbf{M}_1(\varepsilon) \mathbf{F} \mathbf{M}_M \mathbf{I}_{in} = \\
&= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma) \mathbf{M}_2(-\phi) \mathbf{M}_1(-\varepsilon) \mathbf{M}_M \mathbf{M}_M \mathbf{F} \mathbf{I}_{in} = \\
&= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma) \mathbf{M}_2(-\phi) \mathbf{M}_1(-\varepsilon) \mathbf{F} \mathbf{I}_{in}
\end{aligned} \tag{S.6.3.3}$$

where a mirror is additionally placed behind the emitter optics, only the rotation angles between these two elements are inversed, because $\mathbf{M}_M \mathbf{M}_M = \mathbf{1}$, and the circular polarisation is not changed.

Real mirrors are usually dielectric or metal surfaces which can exhibit considerable phase retardation and diattenuation under oblique incident angles. For incident angles smaller than the Brewster angle the phase changes for p- (parallel) and s- (perpendicular) polarised light are in the same direction.

S.7 Standard measurement signals

S.7.1 Lidar signal with rotational error before the polarising beam-splitter

General case with arbitrary laser input and emitter optics $\mathbf{I}_E = \mathbf{M}_E \mathbf{I}_L$ and $\mathbf{R}(\varepsilon, h)$ from Eq. (S.10.15.1):

$$\langle \mathbf{A}_S | = \langle \mathbf{M}_S \mathbf{R}_y | \mathbf{R}(\varepsilon) \mathbf{M}_h = T_S \langle 1 \quad yD_S \quad 0 \quad 0 | \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\varepsilon} & -hs_{2\varepsilon} & 0 \\ 0 & s_{2\varepsilon} & hc_{2\varepsilon} & 0 \\ 0 & 0 & 0 & h \end{pmatrix} = \tag{S.7.1.1}$$

$$= T_S \langle 1 \quad c_{2\varepsilon} yD_S \quad -hs_{2\varepsilon} yD_S \quad 0 |$$

Analyser vector from Eq. (S.7.1.1) and input Stokes vector from (E.31) yield

$$\begin{aligned}
&\frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{R}(\varepsilon) \mathbf{M}_h | \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E \mathbf{I}_L \rangle}{T_S T_{rot} T_O F_{11} T_E I_L} = \\
&= \left\langle \begin{array}{c} 1 \\ c_{2\varepsilon} yD_S \\ -hs_{2\varepsilon} yD_S \\ 0 \end{array} \left\| \begin{array}{c} i_E + D_O a (c_{2\gamma} q_E - s_{2\gamma} u_E) \\ c_{2\gamma} D_O i_E + a q_E - s_{2\gamma} [W_O a (s_{2\gamma} q_E + c_{2\gamma} u_E) + Z_O s_O (1-2a) v_E] \\ s_{2\gamma} D_O i_E - a u_E + c_{2\gamma} [W_O a (s_{2\gamma} q_E + c_{2\gamma} u_E) + Z_O s_O (1-2a) v_E] \\ Z_O s_O a (s_{2\gamma} q_E + c_{2\gamma} u_E) + Z_O c_O (1-2a) v_E \end{array} \right. \right\rangle = \\
&= (1 + yD_S D_O c_{2\gamma+h2\varepsilon}) i_E - yD_S Z_O s_O s_{2\gamma+h2\varepsilon} v_E + \\
&+ a \left\{ D_O (c_{2\gamma} q_E - s_{2\gamma} u_E) + yD_S [(c_{2\varepsilon} q_E + hs_{2\varepsilon} u_E) - s_{2\gamma+h2\varepsilon} (W_O (s_{2\gamma} q_E + c_{2\gamma} u_E) - 2Z_O s_O v_E)] \right\}
\end{aligned} \tag{S.7.1.2}$$

$$\gamma = 0 \Rightarrow$$

$$\begin{aligned}
&\frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{R}(\varepsilon) \mathbf{M}_h | \mathbf{M}_O(0) \mathbf{F}(a) \mathbf{M}_E \mathbf{I}_L \rangle}{T_S T_{rot} T_O F_{11} T_E I_L} = \\
&= (1 + yD_S D_O c_{h2\varepsilon}) i_E - yD_S Z_O s_O s_{h2\varepsilon} v_E + a \left\{ (D_O + yD_S c_{2\varepsilon}) q_E + yD_S s_{h2\varepsilon} Z_O (c_O u_E + 2s_O v_E) \right\}
\end{aligned} \tag{S.7.1.3}$$

$$\gamma = \varepsilon = 0 \Rightarrow$$

$$\frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{R}(0) \mathbf{M}_h | \mathbf{M}_O(0) \mathbf{F}(a) \mathbf{M}_E \mathbf{I}_L \rangle}{T_S T_{rot} T_O F_{11} T_E I_L} = (1 + yD_S D_O) i_E + a(D_O + yD_S) q_E \quad (\text{S.7.1.4})$$

$$\gamma = -h\varepsilon \Rightarrow$$

$$\frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{R}(\varepsilon) \mathbf{M}_h | \mathbf{M}_O(-h\varepsilon) \mathbf{F}(a) \mathbf{M}_E \mathbf{I}_L \rangle}{T_S T_{rot} T_O F_{11} T_E I_L} = (1 + yD_S D_O) i_E + a(D_O + yD_S) (c_{2\gamma} q_E - s_{2\gamma} u_E) \quad (\text{S.7.1.5})$$

With horizontal linearly polarised emitter input Stokes vector \mathbf{I}_E

$$i_E = 1, q_E = 1, u_E = v_E = 0 \Rightarrow$$

$$\frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{R}(\varepsilon) \mathbf{M}_h | \mathbf{M}_O(\gamma) \mathbf{F}(a) | 1 \ 1 \ 0 \ 0 \rangle}{T_S T_{rot} T_O F_{11} T_E I_L} = \quad (\text{S.7.1.6})$$

$$= 1 + yD_S D_O c_{2\gamma+h2\varepsilon} + a \left\{ D_O c_{2\gamma} + yD_S (c_{2\varepsilon} - s_{2\gamma+h2\varepsilon} s_{2\gamma} W_O) \right\}$$

With rotated, linearly polarised laser and emitter optics

$$q_L = 1, u_L = 0, v_L = 0 \Rightarrow$$

$$\frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{R}(\varepsilon) | \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha) \rangle}{T_S T_O F_{11} T_E I_L} =$$

$$= 1 + D_E c_{2\alpha-2\beta} + yD_S \left\{ D_O c_{2\varepsilon+2\gamma} (1 + D_E c_{2\alpha-2\beta}) + Z_O s_O s_{2\varepsilon+2\gamma} Z_E s_E s_{2\alpha-2\beta} \right\} +$$

$$+ a \left\{ \begin{aligned} & D_O \left[c_{2\gamma} (c_{2\beta} D_E + c_{2\alpha} + s_{2\beta} W_E s_{2\alpha-2\beta}) - s_{2\gamma} (s_{2\beta} D_E + s_{2\alpha} - c_{2\beta} W_E s_{2\alpha-2\beta}) \right] + \\ & \left[c_{2\varepsilon} (c_{2\beta} D_E + c_{2\alpha} + s_{2\beta} W_E s_{2\alpha-2\beta}) + s_{2\varepsilon} (s_{2\beta} D_E + s_{2\alpha} - c_{2\beta} W_E s_{2\alpha-2\beta}) \right] - \\ & + yD_S \left[\begin{aligned} & -s_{2\varepsilon+2\gamma} \left(W_O (s_{2\gamma} (c_{2\beta} D_E + c_{2\alpha} + s_{2\beta} W_E s_{2\alpha-2\beta}) + c_{2\gamma} (s_{2\beta} D_E + s_{2\alpha} - c_{2\beta} W_E s_{2\alpha-2\beta})) + \right. \\ & \left. + 2Z_O s_O Z_E s_E s_{2\alpha-2\beta} \right) \end{aligned} \right] \end{aligned} \right\} =$$

$$= (1 + c_{2\gamma+2\varepsilon} yD_S D_O) (1 + c_{2\alpha-2\beta} D_E) + s_{2\gamma+2\varepsilon} s_{2\alpha-2\beta} yD_S Z_O s_O Z_E s_E +$$

$$+ a \left\{ \begin{aligned} & D_O (c_{2\alpha-2\gamma} + c_{2\beta-2\gamma} D_E + s_{2\beta+2\gamma} s_{2\alpha-2\beta} W_E) + \\ & \left[(c_{2\alpha-2\varepsilon} + c_{2\beta-2\varepsilon} D_E + s_{2\beta-2\varepsilon} s_{2\alpha-2\beta} W_E) - \right. \\ & \left. + yD_S \left[\begin{aligned} & -s_{2\gamma+2\varepsilon} W_O (s_{2\alpha+2\gamma} + s_{2\beta+2\gamma} D_E - c_{2\beta-2\gamma} s_{2\alpha-2\beta} W_E) - \\ & -s_{2\gamma+2\varepsilon} s_{2\alpha-2\beta} 2Z_O s_O Z_E s_E \end{aligned} \right] \right] \end{aligned} \right\} \quad (\text{S.7.1.7})$$

$$q_L = 1, u_L = 0, v_L = 0 \wedge \gamma = 0 \Rightarrow$$

$$\begin{aligned} & \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{R}(\varepsilon) \parallel \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha) \rangle}{T_S T_O F_{11} T_E I_L} = \\ & = (1 + c_{2\varepsilon} y D_S D_O) (1 + c_{2\alpha-2\beta} D_E) + s_{2\varepsilon} s_{2\alpha-2\beta} y D_S Z_O S_O Z_E S_E + \\ & \quad + a \left\{ \begin{aligned} & D_O (c_{2\alpha} + c_{2\beta} D_E + s_{2\beta} s_{2\alpha-2\beta} W_E) + \\ & \left[\begin{aligned} & (c_{2\alpha-2\varepsilon} + c_{2\beta-2\varepsilon} D_E + s_{2\beta-2\varepsilon} s_{2\alpha-2\beta} W_E) - \\ & -s_{2\varepsilon} W_O (s_{2\alpha} + s_{2\beta} D_E - c_{2\beta} s_{2\alpha-2\beta} W_E) - \\ & -s_{2\varepsilon} s_{2\alpha-2\beta} 2Z_O S_O Z_E S_E \end{aligned} \right] \end{aligned} \right\} \end{aligned} \quad (\text{S.7.1.8})$$

$$q_L = 1, u_L = 0, v_L = 0, \varepsilon = -\gamma \Rightarrow$$

$$\begin{aligned} & \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{R}(-\gamma) \parallel \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha) \rangle}{T_S T_O F_{11} T_E I_L} = \\ & = (1 + y D_S D_O) (1 + c_{2\alpha-2\beta} D_E) + \\ & \quad + a \left\{ D_O (c_{2\alpha-2\gamma} + c_{2\beta-2\gamma} D_E + s_{2\beta+2\gamma} s_{2\alpha-2\beta} W_E) + y D_S (c_{2\alpha+2\gamma} + c_{2\beta+2\gamma} D_E + s_{2\beta+2\gamma} s_{2\alpha-2\beta} W_E) \right\} \end{aligned} \quad (\text{S.7.1.9})$$

$$q_L = 1, u_L = 0, v_L = 0, \varepsilon = \gamma = 0 \Rightarrow$$

$$\begin{aligned} & \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{R}(\varepsilon) \parallel \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) \mathbf{I}_L(\alpha) \rangle}{T_S T_O F_{11} T_E I_L} = \\ & = (1 + y D_S D_O) (1 + c_{2\alpha-2\beta} D_E) + a (D_O + y D_S) (c_{2\alpha} + c_{2\beta} D_E + s_{2\beta} s_{2\alpha-2\beta} W_E) \end{aligned} \quad (\text{S.7.1.10})$$

S.7.2 Lidar signal with rotational error before the receiving optics

With Eq. (D.7) for the analyser part and (E.26) for the general input vector we get

$$\begin{aligned} & \frac{\langle \mathbf{A}_S(y, \gamma) \parallel \mathbf{I}_{in, \varepsilon}(\varepsilon, h, a) \rangle}{T_S T_O T_{rot} F_{11} T_E I_L} = \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \parallel \mathbf{R}(\varepsilon) \mathbf{M}_h \mathbf{F}(a) \mathbf{I}_E \rangle}{T_S T_O T_{rot} F_{11} T_E I_L} = \\ & = \left\langle \begin{array}{c} 1 + y c_{2\gamma} D_O D_S \\ c_{2\gamma} D_O + y D_S (1 - s_{2\gamma}^2 W_O) \\ s_{2\gamma} (D_O + y c_{2\gamma} D_S W_O) \\ -y s_{2\gamma} D_S Z_O S_O \end{array} \parallel \begin{array}{c} i_E \\ a(q_E c_{2\varepsilon} + h u_E s_{2\varepsilon}) \\ a(q_E s_{2\varepsilon} - h u_E c_{2\varepsilon}) \\ (1 - 2a) h v_E \end{array} \right\rangle = \\ & = (1 + y c_{2\gamma} D_O D_S) i_E - y s_{2\gamma} D_S Z_O S_O h v_E + \\ & \quad + a \left\{ \begin{aligned} & D_O [c_{2\gamma-2\varepsilon} q_E - s_{2\gamma-2\varepsilon} h u_E] - y D_S W_O s_{2\gamma} [s_{2\gamma-2\varepsilon} q_E + c_{2\gamma-2\varepsilon} h u_E] + \\ & + y D_S [q_E c_{2\varepsilon} + h u_E s_{2\varepsilon} + 2s_{2\gamma} Z_O S_O h v_E] \end{aligned} \right\} \end{aligned} \quad (\text{S.7.2.1})$$

Comparison with Eq. (69):

$\varepsilon = 0 \Rightarrow$

$$\begin{aligned} & \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mid \mid \mathbf{R}(\varepsilon) \mathbf{M}_h \mathbf{F}(a) \mathbf{I}_E \rangle}{T_S T_O \quad T_{rot} F_{11} T_E I_L} = \\ & = \left(1 + y c_{2\gamma} D_O D_S\right) i_E - y s_{2\gamma} D_S Z_O s_O h v_E + \\ & + a \left\{ D_O \left[c_{2\gamma} q_E - s_{2\gamma} h u_E \right] - y D_S W_O s_{2\gamma} \left[s_{2\gamma} q_E + c_{2\gamma} h u_E \right] + \right. \\ & \left. + y D_S \left[q_E + 2 s_{2\gamma} Z_O s_O h v_E \right] \right\} \end{aligned} \quad (\text{S.7.2.2})$$

$\gamma = 0 \Rightarrow$

$$\frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mid \mid \mathbf{R}(\varepsilon) \mathbf{M}_h \mathbf{F}(a) \mathbf{I}_E \rangle}{T_S T_O \quad T_{rot} F_{11} T_E I_L} = (1 + y D_O D_S) i_E + a (D_O + y D_S) [c_{2\varepsilon} q_E + s_{2\varepsilon} h u_E] \quad (\text{S.7.2.3})$$

S.7.3 Lidar signal with rotational error behind the emitter optics

We get the equation for this case directly from the previous one considering that moving the matrices for the rotational error from before the receiving optics to behind the emitter optics just changes the sign of the angle ε using Eq. (S.6.2)

$$\begin{aligned} & \frac{\langle \mathbf{A}_S(y, \gamma, a) \mid \mid \mathbf{I}_{in, \varepsilon}(\varepsilon, \mathbf{h}) \rangle}{T_O T_S F_{11} \quad T_{rot} T_E I_L} = \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{F}(a) \mid \mid \mathbf{R}(\varepsilon) \mathbf{M}_h \mathbf{I}_E \rangle}{T_O T_S F_{11} \quad T_{rot} T_E I_L} = \\ & = \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) \mathbf{R}(-\varepsilon) \mathbf{M}_h \mid \mid \mathbf{F}(a) \mathbf{I}_E \rangle}{T_O T_S F_{11} T_{rot} T_E I_L} = \frac{\langle \mathbf{A}_S(y, \gamma) \mid \mid \mathbf{I}_{in, \varepsilon}(-\varepsilon, \mathbf{h}, a) \rangle}{T_S T_O \quad T_{rot} F_{11} T_E I_L} \end{aligned} \quad (\text{S.7.3.1})$$

S.8 Attenuated backscatter coefficient

The attenuated backscatter coefficient F_{11} can be derived from the transmitted signal I_T :

$$\begin{aligned} \eta_T T_T T_O F_{11} T_E I_L & = \frac{I_T}{G_T + a H_T} = \frac{I_T}{G_T + \frac{\delta^* G_T - G_R}{H_R - \delta^* H_T} H_T} = \frac{(H_R - \delta^* H_T) I_T}{(H_R - \delta^* H_T) G_T + (\delta^* G_T - G_R) H_T} \\ & = \frac{(H_R - \delta^* H_T) I_T}{H_R G_T - H_T G_R} = \frac{\left(H_R - \frac{1}{\eta} \frac{I_R}{I_T} H_T \right) I_T}{H_R G_T - H_T G_R} = \frac{H_R I_T - \frac{1}{\eta} I_R H_T}{H_R G_T - H_T G_R} \end{aligned} \quad (\text{S.8.1})$$

With $\eta = \eta_R T_R / \eta_T T_T$ we get the attenuated backscatter coefficient

$$F_{11} = \frac{1}{\eta_T T_T T_O I_L} \frac{H_R I_T - \frac{\eta_T T_T}{\eta_R T_R} H_T I_R}{H_R G_T - H_T G_R} = \frac{1}{T_O I_L} \frac{H_R \frac{I_T}{\eta_T T_T} - H_T \frac{I_R}{\eta_R T_R}}{H_R G_T - H_T G_R} \quad (\text{S.8.2})$$

S.9 Rayleigh calibration

Calibration in ranges with presumably known aerosol depolarisation:

With some lidar systems the calibration factor is determined in a measurement range with known volume linear depolarisation ratio δ , for example in clean air δ^m . Assuming, for the sake of simplicity, an ideal PBS (see Eq. (28)), we get with Eq. (26) for the calibration factor in clean air η^m

$$\delta^*(0^\circ) = \frac{1}{\eta} \frac{I_R}{I_T}(0^\circ) \Rightarrow \eta^m = \frac{1}{\delta^m} \frac{I_R}{I_T}(0^\circ) \quad (\text{S.9.1})$$

Assuming further that the errors in I_R and I_T are independent, which could be the case if the background subtraction or non-linearities in analogue signal detection are the main error sources for them, we get as a first estimate for the relative error of the calibration factor

$$\frac{\Delta\eta^m}{\eta^m} = \frac{\Delta\delta^m}{\delta^m} + \frac{\Delta I_R}{I_R} + \frac{\Delta I_T}{I_T} \quad (\text{S.9.2})$$

This error can easily become very large. The linear depolarisation ratio measured in a volume of air molecules δ^m depends on the width of the interference filters in the receiver optics, as they transmit or reject some rotational Raman lines, and on the atmospheric temperature (Behrendt and Nakamura 2002). At 355 nm δ^m can range from about 0.004 to 0.015. Errors in the order of some 10% in δ^m are already possible in case the wavelength dependence of the transmission of the interference filter or its tilt angle in the optical setup are not known accurately. Furthermore, a small contamination of the assumed clean air with strongly depolarising aerosol as Saharan dust or ice particles from sub-visible cirrus can change the volume linear depolarisation ratio dramatically. Assuming, for example, a small backscatter ratio of 1.01 due to particles with $\delta^p = 0.3$ and with $\delta^m = 0.004$, we get a real $\delta = 0.01 * \delta^p + \delta^m = 0.007$ (Biele et al. 2000), which would cause a relative error in the calibration factor η^m of $(0.007-0.004)/0.004 = +75\%$. Better than this "clean" air calibration would even be to use a calibration in a cirrus cloud with δ^p between let's say 0.3 and 0.5, with a resulting calibration factor error of "only" $\pm \frac{1}{2} * (0.5 - 0.3) / (0.5 + 0.3) = \pm 25\%$.

Summary of this chapter: depolarisation calibration with presumably known atmospheric depolarisation can cause very large calibration errors.

S.10 Some Müller matrices

S.10.1 Depolariser

A diagonal depolariser \mathbf{M}_{DD} (Chipman 2009b)

$$\mathbf{M}_{DD} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \quad (\text{S.10.1.1})$$

partially depolarises the incident light depending on its state of polarisation. For atmospheric depolarisation by randomly oriented, nonspherical particles with rotation and reflection symmetry it can be shown that $b = -a$ and $c = (1 - 2a)$ (van de Hulst 1981; Mishchenko and Hovenier 1995; Mishchenko et al. 2002) (see also Sect. 2.1), which results in the backscattering matrix

$$\mathbf{F} = \begin{pmatrix} F_{11} & 0 & 0 & 0 \\ 0 & F_{22} & 0 & 0 \\ 0 & 0 & -F_{22} & 0 \\ 0 & 0 & 0 & F_{44} \end{pmatrix} = F_{11} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1-2a \end{pmatrix} \quad (\text{S.10.1.2})$$

S.10.2 Rotation matrix for various rotation angles

$$\mathbf{R}(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -s_{2\phi} & 0 \\ 0 & s_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{R}(x\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -xs_{2\phi} & 0 \\ 0 & xs_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{S.10.2.1})$$

$$\mathbf{R}(x45^\circ) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}(x, \varepsilon) = \mathbf{R}(x45^\circ + \varepsilon) = \mathbf{R}(x45^\circ) \mathbf{R}(\varepsilon) = \mathbf{R}(\varepsilon) \mathbf{R}(x45^\circ) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -xs_{2\varepsilon} & -xc_{2\varepsilon} & 0 \\ 0 & xc_{2\varepsilon} & -xs_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{S.10.2.2})$$

See Supp. S.10.15. regarding the half-wave plate rotation.

$$\mathbf{R}(90^\circ) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{R}_y = \mathbf{R}(y) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{matrix} \mathbf{R}(y = -1) = \mathbf{R}(90^\circ) \\ \mathbf{R}(y = +1) = \mathbf{R}(0^\circ) \end{matrix} \quad (\text{S.10.2.3})$$

S.10.3 Retarding linear diattenuator

$$\mathbf{M}_O = \mathbf{M}_D \mathbf{M}_{ret} = \mathbf{M}_{ret} \mathbf{M}_D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_o & s_o \\ 0 & 0 & -s_o & c_o \end{pmatrix} T_o \begin{pmatrix} 1 & D_o & 0 & 0 \\ D_o & 1 & 0 & 0 \\ 0 & 0 & Z_o & 0 \\ 0 & 0 & 0 & Z_o \end{pmatrix} = T_o \begin{pmatrix} 1 & D_o & 0 & 0 \\ D_o & 1 & 0 & 0 \\ 0 & 0 & Z_o c_o & Z_o s_o \\ 0 & 0 & -Z_o s_o & Z_o c_o \end{pmatrix} \quad (\text{S.10.3.1})$$

$$D_o = \frac{T_o^p - T_o^s}{T_o^p + T_o^s}, \quad Z_o = \sqrt{1 - D_o^2}, \quad W_o = 1 - Z_o c_o \quad (\text{S.10.3.2})$$

$$c_o = \cos \Delta_o = \cos(\varphi_o^p - \varphi_o^s), \quad s_o = \sin \Delta_o$$

$$-1 \leq D_o \leq +1 \Rightarrow 0 \leq Z_o \leq 1, \quad 0 \leq W_o \leq 2 \quad (\text{S.10.3.3})$$

S.10.4 Rotated, retarding linear diattenuator

$$\begin{aligned}
\mathbf{M}_O(x\phi) &= \mathbf{R}(x\phi)\mathbf{M}_O\mathbf{R}(-x\phi) = \\
&= T_O \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -xs_{2\phi} & 0 \\ 0 & xs_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & D_O & 0 & 0 \\ D_O & 1 & 0 & 0 \\ 0 & 0 & Z_O c_O & Z_O s_O \\ 0 & 0 & -Z_O s_O & Z_O c_O \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & xs_{2\phi} & 0 \\ 0 & -xs_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\
&= T_O \begin{pmatrix} 1 & c_{2\phi}D_O & xs_{2\phi}D_O & 0 \\ c_{2\phi}D_O & 1-s_{2\phi}^2W_O & xs_{2\phi}c_{2\phi}W_O & -xs_{2\phi}Z_Os_O \\ xs_{2\phi}D_O & xs_{2\phi}c_{2\phi}W_O & 1-c_{2\phi}^2W_O & c_{2\phi}Z_Os_O \\ 0 & xs_{2\phi}Z_Os_O & -c_{2\phi}Z_Os_O & Z_Oc_O \end{pmatrix}
\end{aligned} \tag{S.10.4.1}$$

S.10.5 Rotated, retarding linear diattenuator mirror

$$\begin{aligned}
\mathbf{M}_{MO}(x\phi) &= \mathbf{R}(x\phi)\mathbf{M}_{MO}\mathbf{R}(x\phi) = \\
&= T_O \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -xs_{2\phi} & 0 \\ 0 & xs_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & D_O & 0 & 0 \\ D_O & 1 & 0 & 0 \\ 0 & 0 & -Z_O c_O & -Z_O s_O \\ 0 & 0 & Z_O s_O & -Z_O c_O \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -xs_{2\phi} & 0 \\ 0 & xs_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\
&= T_O \begin{pmatrix} 1 & c_{2\phi}D_O & -xs_{2\phi}D_O & 0 \\ c_{2\phi}D_O & 1-s_{2\phi}^2W_O & -xs_{2\phi}c_{2\phi}W_O & xs_{2\phi}Z_Os_O \\ xs_{2\phi}D_O & xs_{2\phi}c_{2\phi}W_O & -(1-c_{2\phi}^2W_O) & -c_{2\phi}Z_Os_O \\ 0 & xs_{2\phi}Z_Os_O & c_{2\phi}Z_Os_O & -Z_Oc_O \end{pmatrix} = \\
&= T_O \begin{pmatrix} 1 & c_{2\phi}D_O & xs_{2\phi}D_O & 0 \\ c_{2\phi}D_O & 1-s_{2\phi}^2W_O & xs_{2\phi}c_{2\phi}W_O & -xs_{2\phi}Z_Os_O \\ xs_{2\phi}D_O & xs_{2\phi}c_{2\phi}W_O & 1-c_{2\phi}^2W_O & c_{2\phi}Z_Os_O \\ 0 & xs_{2\phi}Z_Os_O & -c_{2\phi}Z_Os_O & Z_Oc_O \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned} \tag{S.10.5.1}$$

S.10.6 $\pm 45^\circ$ rotated retarding linear diattenuator including error ε

$$\begin{aligned}
\mathbf{M}_O(x45^\circ + \varepsilon) &= \mathbf{R}(x45^\circ + \varepsilon)\mathbf{M}_O\mathbf{R}(-x45^\circ - \varepsilon) = \\
&= T_O \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -xs_{2\varepsilon} & -xc_{2\varepsilon} & 0 \\ 0 & xc_{2\varepsilon} & -xs_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & D_O & 0 & 0 \\ D_O & 1 & 0 & 0 \\ 0 & 0 & Z_O c_O & Z_O s_O \\ 0 & 0 & -Z_O s_O & Z_O c_O \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -xs_{2\varepsilon} & xc_{2\varepsilon} & 0 \\ 0 & -xc_{2\varepsilon} & -xs_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\
&= T_O \begin{pmatrix} 1 & -xs_{2\varepsilon}D_O & xc_{2\varepsilon}D_O & 0 \\ -xs_{2\varepsilon}D_O & 1-c_{2\varepsilon}^2W_O & -s_{2\varepsilon}c_{2\varepsilon}W_O & -xc_{2\varepsilon}Z_Os_O \\ xc_{2\varepsilon}D_O & -s_{2\varepsilon}c_{2\varepsilon}W_O & 1-s_{2\varepsilon}^2W_O & -xs_{2\varepsilon}Z_Os_O \\ 0 & xc_{2\varepsilon}Z_Os_O & xs_{2\varepsilon}Z_Os_O & Z_Oc_O \end{pmatrix}
\end{aligned} \tag{S.10.6.1}$$

S.10.7 $\pm 45^\circ$ rotated retarding linear diattenuator

$$\mathbf{M}_O(x45^\circ) = T_O \begin{pmatrix} 1 & 0 & xD_O & 0 \\ 0 & Z_O c_O & 0 & -xZ_O s_O \\ xD_O & 0 & 1 & 0 \\ 0 & xZ_O s_O & 0 & Z_O c_O \end{pmatrix} \quad (\text{S.10.7.1})$$

S.10.8 Rotated, ideal linear polariser and analyser

Note: without absorption $T_P = 0.5$.

$$D_P = 1, Z_P = 0, W_P = 1 \Rightarrow$$

$$\frac{\mathbf{M}_P(0^\circ)}{T_P} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{vmatrix} 1 & & & 1 \\ & 1 & & \\ & & & 1 \\ 0 & & & 0 \end{vmatrix} \quad (\text{S.10.8.1})$$

$$\begin{aligned} \frac{\mathbf{M}_P(x\phi)}{T_P} &= \frac{\mathbf{R}(x\phi) \mathbf{M}_P \mathbf{R}(-x\phi)}{T_P} = \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -xs_{2\phi} & 0 \\ 0 & xs_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{vmatrix} 1 & & & 1 \\ & 1 & & \\ & & & 1 \\ 0 & & & 0 \end{vmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & xs_{2\phi} & 0 \\ 0 & -xs_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \end{aligned} \quad (\text{S.10.8.2})$$

$$= \begin{pmatrix} 1 & c_{2\phi} & xs_{2\phi} & 0 \\ c_{2\phi} & c_{2\phi}^2 & xs_{2\phi}c_{2\phi} & 0 \\ xs_{2\phi} & xs_{2\phi}c_{2\phi} & s_{2\phi}^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{vmatrix} 1 & & & 1 \\ c_{2\phi} & & & c_{2\phi} \\ xs_{2\phi} & & & xs_{2\phi} \\ 0 & & & 0 \end{vmatrix}$$

$$\frac{\mathbf{M}_P(x45^\circ)}{T_P} = \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ x & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{vmatrix} 1 & & & 1 \\ & 0 & & 0 \\ x & & & x \\ 0 & & & 0 \end{vmatrix} \quad (\text{S.10.8.3})$$

$$\frac{\mathbf{M}_P(90^\circ)}{T_P} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{vmatrix} 1 & & & 1 \\ -1 & & & -1 \\ 0 & & & 0 \\ 0 & & & 0 \end{vmatrix} \quad (\text{S.10.8.4})$$

$$D_P = 1, Z_P = 0, W_P = 1 \Rightarrow$$

$$\frac{\mathbf{M}_P(45^\circ - x45^\circ)}{T_P} = \begin{pmatrix} 1 & x & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{vmatrix} 1 & & & 1 \\ x & & & x \\ 0 & & & 0 \\ 0 & & & 0 \end{vmatrix} \quad (\text{S.10.8.5})$$

$$D_p = 1, Z_p = 0, W_p = 1 \Rightarrow$$

$$\mathbf{M}_p(\text{x}45^\circ + \varepsilon) = \mathbf{R}(+\varepsilon) \mathbf{M}_p(\text{x}45^\circ) \mathbf{R}(-\varepsilon) =$$

$$= T_P \begin{pmatrix} 1 & -\text{x}s_{2\varepsilon} & \text{xc}_{2\varepsilon} & 0 \\ -\text{x}s_{2\varepsilon} & s_{2\varepsilon}^2 & -s_{2\varepsilon}c_{2\varepsilon} & 0 \\ \text{xc}_{2\varepsilon} & -s_{2\varepsilon}c_{2\varepsilon} & c_{2\varepsilon}^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \left| \begin{array}{l} 1 \\ -\text{x}s_{2\varepsilon} \\ \text{xc}_{2\varepsilon} \\ 0 \end{array} \right\rangle \left| \begin{array}{l} 1 \\ -\text{x}s_{2\varepsilon} \\ \text{xc}_{2\varepsilon} \\ 0 \end{array} \right\rangle \quad (\text{S.10.8.6})$$

$$D_p = 1, Z_p = 0, W_p = 1 \Rightarrow$$

$$\frac{\mathbf{M}_p(\text{x}45^\circ + \varepsilon) \mathbf{I}_{in}}{T_P \mathbf{I}_{in}} = \left| \begin{array}{l} 1 \\ -\text{x}s_{2\varepsilon} \\ \text{xc}_{2\varepsilon} \\ 0 \end{array} \right\rangle \left| \begin{array}{l} 1 \\ -\text{x}s_{2\varepsilon} \\ \text{xc}_{2\varepsilon} \\ 0 \end{array} \right\rangle \left| \begin{array}{l} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{array} \right\rangle = \left| \begin{array}{l} 1 \\ -\text{x}s_{2\varepsilon} \\ \text{xc}_{2\varepsilon} \\ 0 \end{array} \right\rangle \left[i_{in} - \text{x}(s_{2\varepsilon}q_{in} - c_{2\varepsilon}u_{in}) \right] \quad (\text{S.10.8.7})$$

$$\frac{\mathbf{M}_p(\text{x}45^\circ + \varepsilon) \mathbf{F}(a) \mathbf{I}_{in}}{T_P F_{11} \mathbf{I}_{in}} =$$

$$= \left| \begin{array}{l} 1 \\ -\text{x}s_{2\varepsilon} \\ \text{xc}_{2\varepsilon} \\ 0 \end{array} \right\rangle \left| \begin{array}{l} 1 \\ -\text{x}s_{2\varepsilon} \\ \text{xc}_{2\varepsilon} \\ 0 \end{array} \right\rangle \left| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1-2a \end{pmatrix} \right| \left| \begin{array}{l} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{array} \right\rangle = \left| \begin{array}{l} 1 \\ -\text{x}s_{2\varepsilon} \\ \text{xc}_{2\varepsilon} \\ 0 \end{array} \right\rangle \left[i_{in} - \text{x}a(s_{2\varepsilon}q_{in} + c_{2\varepsilon}u_{in}) \right] \quad (\text{S.10.8.8})$$

$$\frac{\mathbf{F}(a) \mathbf{M}_p(\text{x}45^\circ + \varepsilon) \mathbf{I}_{in}}{T_P F_{11} \mathbf{I}_{in}} =$$

$$= \left| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1-2a \end{pmatrix} \right| \left| \begin{array}{l} 1 \\ -\text{x}s_{2\varepsilon} \\ \text{xc}_{2\varepsilon} \\ 0 \end{array} \right\rangle \left| \begin{array}{l} 1 \\ -\text{x}s_{2\varepsilon} \\ \text{xc}_{2\varepsilon} \\ 0 \end{array} \right\rangle \left| \begin{array}{l} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{array} \right\rangle = \left| \begin{array}{l} 1 \\ -\text{x}as_{2\varepsilon} \\ -\text{x}ac_{2\varepsilon} \\ 0 \end{array} \right\rangle \left[i_{in} - \text{x}(s_{2\varepsilon}q_{in} - c_{2\varepsilon}u_{in}) \right] \quad (\text{S.10.8.9})$$

For the 0° and 90° measurements with a perfect polariser \mathbf{M}_p we get from Eq. (S.10.8.1) and Eq. (S.12.2)

$$\text{for } D_p = 1, Z_p = 0, W_p = 1$$

$$\mathbf{M}_p(-\text{x}45^\circ + 45^\circ + \varepsilon) \mathbf{I}_{in} =$$

$$= T_P \mathbf{I}_{in} \begin{pmatrix} 1 & \text{xc}_{2\varepsilon} & \text{xs}_{2\varepsilon} & 0 \\ \text{xc}_{2\varepsilon} & c_{2\varepsilon}^2 & s_{2\varepsilon}c_{2\varepsilon} & 0 \\ \text{xs}_{2\varepsilon} & s_{2\varepsilon}c_{2\varepsilon} & s_{2\varepsilon}^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \left| \begin{array}{l} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{array} \right\rangle = T_P \mathbf{I}_{in} \left| \begin{array}{l} i_{in} + \text{x}(c_{2\varepsilon}q_{in} - s_{2\varepsilon}u_{in}) \\ \text{xc}_{2\varepsilon}i_{in} + c_{2\varepsilon}^2q_{in} + s_{2\varepsilon}c_{2\varepsilon}u_{in} \\ \text{xs}_{2\varepsilon}i_{in} + s_{2\varepsilon}c_{2\varepsilon}q_{in} + s_{2\varepsilon}^2u_{in} \\ 0 \end{array} \right\rangle = \quad (\text{S.10.8.10})$$

$$= T_P \mathbf{I}_{in} \left| \begin{array}{l} i_{in} + \text{x}(c_{2\varepsilon}q_{in} - s_{2\varepsilon}u_{in}) \\ \text{xc}_{2\varepsilon}i_{in} + c_{2\varepsilon}(c_{2\varepsilon}q_{in} - s_{2\varepsilon}u_{in}) \\ \text{xs}_{2\varepsilon}i_{in} + s_{2\varepsilon}(c_{2\varepsilon}q_{in} - s_{2\varepsilon}u_{in}) \\ 0 \end{array} \right\rangle = T_P \mathbf{I}_{in} \left[i_{in} + \text{x}(c_{2\varepsilon}q_{in} - s_{2\varepsilon}u_{in}) \right] \left| \begin{array}{l} 1 \\ \text{xc}_{2\varepsilon} \\ \text{xs}_{2\varepsilon} \\ 0 \end{array} \right\rangle$$

S.10.9 Two rotated retarding linear diattenuators

$$\begin{aligned}
\mathbf{M}_A(\phi)\mathbf{M}_O(\gamma) &= \\
&= \mathbf{R}(\phi)\mathbf{M}_A\mathbf{R}(-\phi)\mathbf{R}(\gamma)\mathbf{M}_O \quad \mathbf{R}(-\gamma) = \\
&= \mathbf{R}(\phi)\mathbf{M}_A \quad \mathbf{R}(\gamma - \phi)\mathbf{M}_O \quad \mathbf{R}(-\gamma) = \\
&= \mathbf{R}(\phi)\mathbf{M}_A \quad \mathbf{R}(\gamma - \phi)\mathbf{M}_O \quad \mathbf{R}(-\gamma)\mathbf{R}(\phi)\mathbf{R}(-\phi) = \\
&= \mathbf{R}(\phi)\mathbf{M}_A \quad \mathbf{R}(\gamma - \phi)\mathbf{M}_O \quad \mathbf{R}(\phi - \gamma)\mathbf{R}(-\phi) = \\
&= \mathbf{R}(\phi)\mathbf{M}_A \quad \mathbf{M}_O(\gamma - \phi)\mathbf{R}(-\phi)
\end{aligned} \tag{S.10.9.1}$$

$$\begin{aligned}
\frac{\mathbf{M}_A(\phi)\mathbf{M}_O(\gamma)}{T_A T_O} &= \\
&= \begin{pmatrix} 1 & c_{2\phi}D_A & s_{2\phi}D_A & 0 \\ c_{2\phi}D_A & 1 - s_{2\phi}^2W_A & s_{2\phi}c_{2\phi}W_A & -s_{2\phi}Z_A S_A \\ s_{2\phi}D_A & s_{2\phi}c_{2\phi}W_A & 1 - c_{2\phi}^2W_A & c_{2\phi}Z_A S_A \\ 0 & s_{2\phi}Z_A S_A & -c_{2\phi}Z_A S_A & Z_A c_A \end{pmatrix} \begin{pmatrix} 1 & c_{2\gamma}D_O & s_{2\gamma}D_O & 0 \\ c_{2\gamma}D_O & 1 - s_{2\gamma}^2W_O & s_{2\gamma}c_{2\gamma}W_O & -s_{2\gamma}Z_O S_O \\ s_{2\gamma}D_O & s_{2\gamma}c_{2\gamma}W_O & 1 - c_{2\gamma}^2W_O & c_{2\gamma}Z_O S_O \\ 0 & s_{2\gamma}Z_O S_O & -c_{2\gamma}Z_O S_O & Z_O c_O \end{pmatrix}
\end{aligned} \tag{S.10.9.2}$$

The first row vector of Eq. (S.10.9.2)

$$\begin{aligned}
\frac{\langle \mathbf{M}_A(\phi)\mathbf{M}_O(\gamma) \rangle}{T_A T_O} &= \\
&= \begin{pmatrix} 1 + c_{2\gamma-2\phi}D_A D_O \\ c_{2\gamma}D_O + (1 - s_{2\gamma}^2W_O)c_{2\phi}D_A + s_{2\gamma}c_{2\gamma}W_O s_{2\phi}D_A \\ s_{2\gamma}(D_O + c_{2\gamma}W_O c_{2\phi}D_A) + (1 - c_{2\gamma}^2W_O)s_{2\phi}D_A \\ -s_{2\gamma}Z_O S_O c_{2\phi}D_A + c_{2\gamma}Z_O S_O s_{2\phi}D_A \end{pmatrix} = \begin{pmatrix} 1 + c_{2\gamma-2\phi}D_A D_O \\ c_{2\gamma}D_O + (c_{2\phi} - s_{2\gamma}s_{2\gamma-2\phi}W_O)D_A \\ s_{2\gamma}D_O + (s_{2\phi} + c_{2\gamma}s_{2\gamma-2\phi}W_O)D_A \\ -s_{2\gamma-2\phi}Z_O S_O D_A \end{pmatrix}
\end{aligned} \tag{S.10.9.3}$$

S.10.10 Cleaned analyser (polarising beam-splitter with additional polarising sheet filters)

The intensity transmission of analysers is proportional to a certain state of polarisation before the analyser, with arbitrary state of polarisation behind, while the output of polarisers is a certain state of polarisation regardless which state of polarisation exists before the polariser (Lu and Chipman 1996). Here we use a polarising sheet filter, which is a depolarising analyser and a depolarising polariser at the same time, to get rid of the cross talk of the polarising beam-splitter. The combined matrix of a polarising sheet filter \mathbf{M}_A behind the polarising beam-splitter \mathbf{M}_S is again the matrix of a retarding linear diattenuator, which we call a cleaned polarising beam-splitter. If \mathbf{M}_A is rotated (misaligned) by ϕ we get from Eq. (S.10.9.3)

$$\begin{aligned}
\gamma = 0 &\Rightarrow \\
\frac{\langle \mathbf{M}_A(\phi)\mathbf{M}_S(0) \rangle}{T_A T_S} &= \langle 1 + c_{2\phi}D_A D_S \quad D_S + c_{2\phi}D_A \quad s_{2\phi}D_A Z_S c_S \quad s_{2\phi}D_A Z_S S_S \rangle
\end{aligned} \tag{S.10.10.1}$$

Transmitted part:

$$\gamma = 0, \phi = 0 \Rightarrow \frac{\langle \mathbf{M}_A(0) \mathbf{M}_T(0) \rangle}{T_A T_T} = \langle 1 + D_A D_T \quad D_T + D_A \quad 0 \quad 0 \rangle \quad (\text{S.10.10.2})$$

Reflected part:

$$\gamma = 0, \phi = 90^\circ \Rightarrow \frac{\langle \mathbf{M}_A(90^\circ) \mathbf{M}_S(0) \rangle}{T_A T_S} = \langle 1 - D_A D_S \quad D_S - D_A \quad 0 \quad 0 \rangle \quad (\text{S.10.10.3})$$

Typically the manufacturers' terminology for p- and s-polarised transmission is k_l and k_2 , respectively, as Eq. (S.10.10.4), and their specifications for polarising sheet filters are the transmission of two crossed filters (T_{cross}) Eq. (S.10.10.6) and that of two parallel filters ($T_{parallel}$) Eq. (S.10.10.7) of the same type.

$$T_A^p = k_1 \quad \text{and} \quad T_A^s = k_2 \quad \Rightarrow \quad (\text{S.10.10.4})$$

$$D_A = \frac{k_1 - k_2}{k_1 + k_2}, \quad Z_A = \frac{2\sqrt{k_1 k_2}}{k_1 + k_2}, \quad T_A = \frac{k_1 + k_2}{2} \quad (\text{S.10.10.5})$$

$$T_{cross} = H_{90} = k_1 k_2 = T_A \sqrt{1 - D_A^2} \quad (\text{S.10.10.6})$$

$$T_{parallel} = H_0 = 0.5(k_1^2 + k_2^2) = T_A \sqrt{1 + D_A^2} \quad (\text{S.10.10.7})$$

For the extinction ratio ρ (Eq. S.10.10.8, see Bennett (2009a), Sect. 12.4) and its inverse, i.e. the contrast ratio or transmission ratio, different definitions, as in Eq. (S.10.10.9), can be found in manufacturers' descriptions, which is sometimes confusing. However, usually $k_2 \ll k_1$, and the given extinction ratios are then to be understood as "on the order of", irrespective of the used formula.

$$\rho = \frac{k_2}{k_1} = \frac{T_A^s}{T_A^p} = \frac{1 - D_A}{1 + D_A} \quad (\text{S.10.10.8})$$

$$k_2 \ll k_1 \Rightarrow$$

$$\frac{T_{cross}}{T_{parallel}} = \frac{H_{90}}{H_0} = \frac{k_1 k_2}{0.5(k_1^2 + k_2^2)} \approx 2\rho \quad (\text{S.10.10.9})$$

$$\begin{aligned} D_T^\# &= \frac{D_T + D_A}{1 + D_T D_A} = \frac{T_T^p k_1 - T_T^s k_2}{T_T^p k_1 + T_T^s k_2}, & D_R^\# &= \frac{D_R - D_A}{1 - D_R D_A} = \frac{T_R^p k_2 - T_R^s k_1}{T_R^p k_2 + T_R^s k_1} \\ T_T^\# &= T_T T_A (1 + D_T D_A) = 0.5(T_T^p k_1 + T_T^s k_2), & T_R^\# &= T_R T_A (1 - D_R D_A) = 0.5(T_R^p k_2 + T_R^s k_1) \\ Z_T^\# &= \frac{Z_T Z_A}{1 + D_T D_A} = \frac{2\sqrt{T_T^p k_1 T_T^s k_2}}{T_T^p k_1 + T_T^s k_2}, & Z_R^\# &= \frac{Z_R Z_A}{1 - D_R D_A} = \frac{2\sqrt{T_R^s k_1 T_R^p k_2}}{T_R^p k_2 + T_R^s k_1} \end{aligned} \quad (\text{S.10.10.10})$$

For an ideal (cleaned) analyser \mathbf{M}_A with total extinction ($k_2 = 0$) we get from Eqs. (S.10.10.3) and (S.10.10.10)

with $k_2 = 0, D_A = 1, Z_A = 0 \Rightarrow$

$$T_T^\# = 0.5T_T^p k_1, \quad T_R^\# = 0.5T_R^s k_1, \quad D_T^\# = +1, \quad D_R^\# = -1, \quad Z_S^\# = 0 \quad (\text{S.10.10.11})$$

$$\eta = \frac{\eta_R T_R^\#}{\eta_T T_T^\#} = \frac{\eta_R T_R^s}{\eta_T T_T^p} \quad (\text{S.10.10.12})$$

General:

$$\langle \mathbf{M}_{21} | \equiv \langle \mathbf{M}_2(0^\circ) \mathbf{M}_1(0^\circ) | = T_2 T_1 \langle 1 + D_2 D_1 \quad D_2 + D_1 \quad 0 \quad 0 | = T_{21} \langle 1 \quad D_{21} \quad 0 \quad 0 |$$

$$D_1 = \frac{T_1^p - T_1^s}{T_1^p + T_1^s}$$

$$T_1 = 0.5(T_1^p + T_1^s)$$

$$D_{21} = \frac{D_2 + D_1}{1 + D_2 D_1} = \frac{T_2^p T_1^p - T_2^s T_1^s}{T_2^p T_1^p + T_2^s T_1^s}, \quad (\text{S.10.10.13})$$

$$T_{21} = T_2 T_1 (1 + D_2 D_1) = 0.5(T_2^p T_1^p + T_2^s T_1^s)$$

$$Z_{21} = \frac{Z_2 Z_1}{1 + D_2 D_1} = \frac{2\sqrt{T_2^p T_1^p T_2^s T_1^s}}{T_2^p T_1^p + T_2^s T_1^s}$$

$$1 - D_1 = \frac{T_1^s}{T_1}, \quad 1 + D_1 = \frac{T_1^p}{T_1} \quad (\text{S.10.10.14})$$

$$D_{S_{yO}} = \frac{D_O + yD_S}{1 + yD_S D_O} = \frac{(1+y)[T_O^p T_S^p - T_O^s T_S^s] + (1-y)[T_O^p T_S^s - T_O^s T_S^p]}{(1+y)[T_O^p T_S^p + T_O^s T_S^s] + (1-y)[T_O^p T_S^s + T_O^s T_S^p]}$$

$$T_{S_{yO}} = T_S T_O (1 + yD_S D_O) = 0.25 \left\{ (1+y)[T_O^p T_S^p + T_O^s T_S^s] + (1-y)[T_O^p T_S^s + T_O^s T_S^p] \right\}$$

$$y = +1 \Rightarrow D_{S_{+O}} = \frac{T_O^p T_S^p - T_O^s T_S^s}{T_O^p T_S^p + T_O^s T_S^s}, \quad T_{S_{+O}} = 0.5(T_O^p T_S^p + T_O^s T_S^s), \quad T_{S_{+O}}^p = T_O^p T_S^p, \quad T_{S_{+O}}^s = T_O^s T_S^s$$

$$y = -1 \Rightarrow D_{S_{-O}} = \frac{T_O^p T_S^s - T_O^s T_S^p}{T_O^p T_S^s + T_O^s T_S^p}, \quad T_{S_{-O}} = 0.5(T_O^p T_S^s + T_O^s T_S^p), \quad T_{S_{-O}}^p = T_O^p T_S^s, \quad T_{S_{-O}}^s = T_O^s T_S^p$$

(S.10.10.15)

$$D_S = \pm 1 \wedge y = \pm 1 \Rightarrow D_S^2 = y^2 = 1 \Rightarrow$$

$$D_{S_{yO}} = \frac{D_O + yD_S}{1 + yD_S D_O} = \frac{1}{yD_S} \frac{yD_S D_O + 1}{1 + yD_S D_O} = \frac{1}{yD_S} = yD_S \quad (\text{S.10.10.16})$$

$$1 - D_{S_{yO}} = \frac{T_{S_{yO}}^s}{T_{S_{yO}}}, \quad 1 + D_{S_{yO}} = \frac{T_{S_{yO}}^p}{T_{S_{yO}}} \quad (\text{S.10.10.17})$$

S.10.11 Retarder

A retarder is a retarding linear diattenuator (Suppl. S.10.3ff) without diattenuation (see Chipman (2009b)):

$D_o = 0, Z_o = 1$ (without absorption $T_o = 1$) \Rightarrow

$$M_{ret} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_o & s_o \\ 0 & 0 & -s_o & c_o \end{pmatrix} \quad (\text{S.10.11.1})$$

S.10.12 Rotated retarder

Rotated retarder with

$$D_o = 0 \Rightarrow Z_o = \sqrt{1 - D_o^2} = 1, W_o = 1 - Z_o c_o = 1 - c_o \Rightarrow$$

$$\mathbf{M}_{Ret}(x\phi)/T_{Ret} = \mathbf{R}(x\phi) \mathbf{M}_o \mathbf{R}(-x\phi) =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - s_{2\phi}^2(1 - c_o) & xs_{2\phi}c_{2\phi}(1 - c_o) & -xs_{2\phi}s_o \\ 0 & xs_{2\phi}c_{2\phi}(1 - c_o) & 1 - c_{2\phi}^2(1 - c_o) & c_{2\phi}s_o \\ 0 & xs_{2\phi}s_o & -c_{2\phi}s_o & c_o \end{pmatrix} \quad (\text{S.10.12.1})$$

with

$$1 - s_{2\phi}^2(1 - c_o) = c_{2\phi}^2 + s_{2\phi}^2 c_o$$

$$1 - c_{2\phi}^2(1 - c_o) = s_{2\phi}^2 + c_{2\phi}^2 c_o$$

S.10.13 Rotated $\lambda/2$ plate (HWP)

Retarder Eq.(S.10.12.1) with

$$\Delta_o = 180^\circ \Rightarrow c_o = -1, s_o = 0, D_o = 0, Z_o = \sqrt{1 - D_o^2} = 1, W_o = 1 - Z_o c_o = 2 \quad (\text{S.10.13.1})$$

$$\mathbf{M}_{HW}(\theta)/T_{HW} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - 2s_{2\theta}^2 & 2c_{2\theta}s_{2\theta} & 0 \\ 0 & 2c_{2\theta}s_{2\theta} & 1 - 2c_{2\theta}^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{4\theta} & s_{4\theta} & 0 \\ 0 & s_{4\theta} & -c_{4\theta} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathbf{R}(2\theta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{S.10.13.2})$$

The rotation of a $\lambda/2$ plate by ϕ rotates the Stokes vector by twice the rotation ϕ and additionally inverts the circular polarisation component. This is equivalent to a mirror followed by a rotation of the coordinate system by 2θ . Please note, that the rotator and mirror matrices don't commute (compare Eqs. (S.6.2.1) ff).

$\lambda/2$ -retarder Eq.(S.10.12.1) at 0° and 22.5° with phase shift error 2ω

$$D_o = 0 \Rightarrow Z_o = \sqrt{1 - D_o^2} = 1$$

$$\Delta_o = \pi + 2\omega \Rightarrow c_o \approx -1 + 2\omega^2, s_o \approx 2\omega \Rightarrow W_o = 1 - Z_o c_o \approx 2 - 2\omega^2 \quad (\text{S.10.13.3})$$

$$\phi = 22.5^\circ \Rightarrow s_{2\phi} = c_{2\phi} = \frac{1}{\sqrt{2}}$$

$$\mathbf{M}_{HW}(0^\circ, 2\omega)/T_{HW} \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1+2\omega^2 & 2\omega \\ 0 & 0 & -2\omega & -1+2\omega^2 \end{pmatrix} \quad (\text{S.10.13.4})$$

$$\mathbf{M}_{HW}(x22.5^\circ, 2\omega)/T_{HW} \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^2 & x(1-\omega^2) & -x\sqrt{2}\omega \\ 0 & x(1-\omega^2) & \omega^2 & \sqrt{2}\omega \\ 0 & x\sqrt{2}\omega & -\sqrt{2}\omega & -1+2\omega^2 \end{pmatrix} \quad (\text{S.10.13.5})$$

$$\begin{aligned} \langle \mathbf{A}_s(y) | \mathbf{M}_{HW}(x22.5^\circ, 2\omega) \approx \\ \approx T_s \left\langle \begin{array}{c} 1 \\ yD_s \\ 0 \\ 0 \end{array} \right| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^2 & x(1-\omega^2) & -x\sqrt{2}\omega \\ 0 & x(1-\omega^2) & \omega^2 & \sqrt{2}\omega \\ 0 & x\sqrt{2}\omega & -\sqrt{2}\omega & -1+2\omega^2 \end{pmatrix} = \left\langle \begin{array}{c} 1 \\ -y\omega^2 D_s \\ -xy(1-\omega^2) D_s \\ xy\sqrt{2}\omega D_s \end{array} \right| = \\ = \left\langle \begin{array}{c} 1 \\ 0 \\ -xyD_s \\ 0 \end{array} \right| + y\omega D_s \left\langle \begin{array}{c} 0 \\ -\omega \\ x\omega \\ x\sqrt{2} \end{array} \right| \end{aligned} \quad (\text{S.10.13.6})$$

S.10.14 Rotated $\lambda/2$ plate (HWP) for $\Delta 90$ -calibration including error ε

$$\begin{aligned} \frac{\mathbf{M}_{HW}(x22.5^\circ + \varepsilon/2)}{T_{HW}} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -xS_{2\varepsilon} & xC_{2\varepsilon} & 0 \\ 0 & xC_{2\varepsilon} & xS_{2\varepsilon} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -xS_{2\varepsilon} & -xC_{2\varepsilon} & 0 \\ 0 & xC_{2\varepsilon} & -xS_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathbf{R}(x45^\circ + \varepsilon) \mathbf{M}_M \end{aligned} \quad (\text{S.10.14.1})$$

S.10.15 Rotation calibrator

The mechanical rotator (Sect. S.10.2) and the $\lambda/2$ -rotator (Sects. S.10.13, S.10.14) can be combined to

$$\frac{\mathbf{M}_{rot}(\phi, h)}{T_{rot}} = \mathbf{R}(\phi) \mathbf{M}_h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -s_{2\phi} & 0 \\ 0 & s_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & h \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -hs_{2\phi} & 0 \\ 0 & s_{2\phi} & hc_{2\phi} & 0 \\ 0 & 0 & 0 & h \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & h \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -hs_{2\phi} & 0 \\ 0 & hs_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & h \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{h2\phi} & -s_{h2\phi} & 0 \\ 0 & s_{h2\phi} & c_{h2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $h = \pm 1$

(S.10.15.1)

where T_{rot} is the transmission of the rotation calibrator for unpolarised light, which equals one for the mechanical rotator. For the mechanical rotator we use $h = +1$, and for the $\lambda/2$ -rotator $h = -1$, and $\phi = 2\theta$ is two times the actual rotation θ of the $\lambda/2$ -plate, as well as ε is two times the actual error angle of the $\lambda/2$ -plate. With Eq. (S.10.15.1) we get Eq. (S.10.15.2) for the rotation calibrator \mathbf{M}_{rot} at $\pm 45^\circ$.

$$\mathbf{M}_{rot}(x45^\circ + \varepsilon, h) / T_{rot} = \mathbf{R}(x45^\circ + \varepsilon) \mathbf{M}_h = \mathbf{R}(x45^\circ) \mathbf{R}(\varepsilon) \mathbf{M}_h =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\varepsilon} & -s_{2\varepsilon} & 0 \\ 0 & s_{2\varepsilon} & c_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & h \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -xs_{2\varepsilon} & -xhc_{2\varepsilon} & 0 \\ 0 & xc_{2\varepsilon} & -xhs_{2\varepsilon} & 0 \\ 0 & 0 & 0 & h \end{pmatrix} \quad (\text{S.10.15.2})$$

with $x, h = \pm 1$

The error rotation ε can be separated as in Eq. (S.10.15.3) using the explanations in Sect. S.6.3 .

$$\mathbf{R}(x45^\circ + \varepsilon) \mathbf{M}_h = \mathbf{R}(x45^\circ) \mathbf{R}(\varepsilon) \mathbf{M}_h = \mathbf{R}(x45^\circ) \mathbf{M}_h \mathbf{R}(h\varepsilon) \quad (\text{S.10.15.3})$$

S.10.16 $\lambda/4$ plate (QWP)

From Eq. (S.10.6.1): QWP without diattenuation, with phase shift error ω .

$$\Delta_{QW} = 90^\circ + \omega \Rightarrow c_Q = -s_\omega, \quad s_Q = c_\omega$$

$$\phi = x45^\circ + \varepsilon \Rightarrow c_{2\phi} \rightarrow -xs_{2\varepsilon}, \quad s_{2\phi} \rightarrow xc_{2\varepsilon}$$

$$D_{QW} = 0, Z_{QW} = 1, W_Q = (1 - c_{QW}) = (1 + s_\omega) \quad (\text{S.10.16.1})$$

$$(1 - c_{2\varepsilon}^2 W_{QW}) = s_{2\varepsilon}^2 + c_{2\varepsilon}^2 c_{QW} = (s_{2\varepsilon}^2 - c_{2\varepsilon}^2 s_\omega)$$

$$(1 - s_{2\varepsilon}^2 W_{QW}) = c_{2\varepsilon}^2 + s_{2\varepsilon}^2 c_{QW} = (c_{2\varepsilon}^2 - s_{2\varepsilon}^2 s_\omega)$$

$$D_{QW} = 0, Z_{QW} = 1, W_{QW} = (1 - c_{2\varepsilon}) = (1 + s_\omega) \Rightarrow$$

$$\frac{\mathbf{M}_{QW}(x45^\circ + \varepsilon, \omega)}{T_{QW}} = \frac{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1 - c_{2\varepsilon}^2 W_{QW}) & -c_{2\varepsilon} s_{2\varepsilon} W_{QW} & -x c_{2\varepsilon} s_{2\varepsilon} \\ 0 & -s_{2\varepsilon} c_{2\varepsilon} W_{QW} & (1 - s_{2\varepsilon}^2 W_{QW}) & -x s_{2\varepsilon} s_{2\varepsilon} \\ 0 & x c_{2\varepsilon} s_{2\varepsilon} & x s_{2\varepsilon} s_{2\varepsilon} & c_{2\varepsilon} \end{pmatrix}}{T_{QW}} = \frac{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s_{2\varepsilon}^2 - c_{2\varepsilon}^2 s_\omega & -s_{2\varepsilon} c_{2\varepsilon} (1 + s_\omega) & -x c_{2\varepsilon} c_\omega \\ 0 & -s_{2\varepsilon} c_{2\varepsilon} (1 + s_\omega) & c_{2\varepsilon}^2 - s_{2\varepsilon}^2 s_\omega & -x s_{2\varepsilon} c_\omega \\ 0 & x c_{2\varepsilon} c_\omega & x s_{2\varepsilon} c_\omega & -s_\omega \end{pmatrix}}{T_{QW}} = \frac{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - c_{2\varepsilon}^2 (1 + s_\omega) & -c_{2\varepsilon} s_{2\varepsilon} (1 + s_\omega) & -x c_{2\varepsilon} c_\omega \\ 0 & -s_{2\varepsilon} c_{2\varepsilon} (1 + s_\omega) & 1 - s_{2\varepsilon}^2 (1 + s_\omega) & -x s_{2\varepsilon} c_\omega \\ 0 & x c_{2\varepsilon} c_\omega & x s_{2\varepsilon} c_\omega & -s_\omega \end{pmatrix}}{T_{QW}} \quad (\text{S.10.16.2})$$

$$D_{QW} = 0, W_Q = (1 + s_\omega), \varepsilon = 0 \Rightarrow$$

$$\frac{\mathbf{M}_{QW}(x45^\circ, \omega)}{T_{QW}} = \frac{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -s_\omega & 0 & -x c_\omega \\ 0 & 0 & 1 & 0 \\ 0 & x c_\omega & 0 & -s_\omega \end{pmatrix}}{T_{QW}} \quad (\text{S.10.16.3})$$

S.10.17 Rotated, ideal $\lambda/4$ plate (QWP)

$$\Delta_o = 90^\circ \Rightarrow c_o = 0, s_o = 1, D_o = 0, Z_o = \sqrt{1 - D_o^2} = 1, W_o = 1 - Z_o c_o = 1 \quad (\text{S.10.17.1})$$

(without absorption $T_{QWP} = 1$)

$$\mathbf{M}_{QW}(\phi) = T_{QW} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi}^2 & s_{2\phi} c_{2\phi} & -s_{2\phi} \\ 0 & s_{2\phi} c_{2\phi} & s_{2\phi}^2 & c_{2\phi} \\ 0 & s_{2\phi} & -c_{2\phi} & 0 \end{pmatrix} \quad (\text{S.10.17.2})$$

$$\mathbf{M}_{QW}(x45^\circ + \varepsilon) = \mathbf{R}(x45^\circ + \varepsilon) \mathbf{M}_{QW} \mathbf{R}(-x45^\circ - \varepsilon) =$$

$$= T_{QW} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s_{2\varepsilon}^2 & -s_{2\varepsilon} c_{2\varepsilon} & -x c_{2\varepsilon} \\ 0 & -s_{2\varepsilon} c_{2\varepsilon} & c_{2\varepsilon}^2 & -x s_{2\varepsilon} \\ 0 & x c_{2\varepsilon} & x s_{2\varepsilon} & 0 \end{pmatrix} \quad (\text{S.10.17.3})$$

$$\frac{\mathbf{M}_{QW}(x45^\circ)}{T_{QW}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x \\ 0 & 0 & 1 & 0 \\ 0 & x & 0 & 0 \end{pmatrix} \quad (\text{S.10.17.4})$$

$$\mathbf{M}_{QW}(0) = T_{QW} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (\text{S.10.17.5})$$

$$x \in \{0, \pm 1\}$$

$$\mathbf{M}_{QW}(x45^\circ) = T_{QW} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-x^2 & 0 & -x \\ 0 & 0 & x^2 & 1-x^2 \\ 0 & x & -(1-x^2) & 0 \end{pmatrix} \quad (\text{S.10.17.6})$$

$$x \in \{0, \pm 1\}$$

$$\mathbf{M}_{QW}(45^\circ - x45^\circ) = T_{QW} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x^2 & 0 & -(1-x^2) \\ 0 & 0 & 1-x^2 & x \\ 0 & 1-x^2 & -x & 0 \end{pmatrix} \quad (\text{S.10.17.7})$$

S.10.18 Circular polariser (CP)

Linear polariser at 0° Eq. (S.10.3.1) and QWP at $z45^\circ$ Eq. (S.10.16.3) (see Chipman (2009a) Chap. 15.26).

$$D_{QW} = 0, \quad Z_{QW} = 1, \quad \Delta_{QW} = 90^\circ + \omega \Rightarrow W_{QW} = (1 + s_\omega) =$$

$$\frac{\mathbf{M}_{QW}(z45^\circ, \omega) \mathbf{M}_P(0^\circ)}{T_{QW} T_P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -s_\omega & 0 & -zc_\omega \\ 0 & 0 & 1 & 0 \\ 0 & zc_\omega & 0 & -s_\omega \end{pmatrix} \begin{pmatrix} 1 & D_P & 0 & 0 \\ D_P & 1 & 0 & 0 \\ 0 & 0 & Z_P c_P & Z_P s_P \\ 0 & 0 & -Z_P s_P & Z_P c_P \end{pmatrix} = \quad (\text{S.10.18.1})$$

$$= \begin{pmatrix} 1 & D_P & 0 & 0 \\ -s_\omega D_P & -s_\omega & zc_\omega Z_P s_P & -zc_\omega Z_P c_P \\ 0 & 0 & Z_P c_P & Z_P s_P \\ zc_\omega D_P & zc_\omega & s_\omega Z_P s_P & -s_\omega Z_P c_P \end{pmatrix}$$

Circular polariser as $\pm 45^\circ$ calibrator with a QWP with phase shift error ω and a real linear polariser

$$\frac{[\mathbf{M}_{QW}(z45^\circ, \omega) \mathbf{M}_P(0^\circ)](x45^\circ + \varepsilon)}{T_{QW}T_P} = \begin{pmatrix} 1 & -xS_{2\varepsilon}D_P & \dots & \dots \\ xS_{2\varepsilon}S_\omega D_P & -S_{2\varepsilon}^2S_\omega + c_{2\varepsilon}(c_{2\varepsilon}c_P + s_{2\varepsilon}zC_\omega S_P)Z_P & \dots & \dots \\ -xC_{2\varepsilon}S_\omega D_P & c_{2\varepsilon}S_{2\varepsilon}S_\omega + c_{2\varepsilon}(s_{2\varepsilon}c_P - c_{2\varepsilon}zC_\omega S_P)Z_P & \dots & \dots \\ zC_\omega D_P & -x(s_{2\varepsilon}zC_\omega + c_{2\varepsilon}S_\omega S_P Z_P) & \dots & \dots \\ \dots & \dots & xC_{2\varepsilon}D_P & 0 \\ \dots & \dots & c_{2\varepsilon}S_{2\varepsilon}S_\omega + s_{2\varepsilon}(c_{2\varepsilon}c_P + s_{2\varepsilon}zC_\omega S_P)Z_P & x(s_{2\varepsilon}zC_\omega c_P - c_{2\varepsilon}S_P)Z_P \\ \dots & \dots & -c_{2\varepsilon}^2S_\omega + s_{2\varepsilon}(s_{2\varepsilon}c_P - c_{2\varepsilon}zC_\omega S_P)Z_P & -x(c_{2\varepsilon}zC_\omega c_P + s_{2\varepsilon}S_P)Z_P \\ \dots & \dots & x(c_{2\varepsilon}zC_\omega - s_{2\varepsilon}S_\omega S_P Z_P) & -s_\omega c_P Z_P \end{pmatrix} \quad (\text{S.10.18.2})$$

Circular polariser with QWP without phase shift error ω and real linear polariser as $\pm 45^\circ$ calibrator

$$\omega = 0 \Rightarrow$$

$$\frac{\mathbf{M}_{CP}(z, 0, x45^\circ + \varepsilon)}{T_{CP}} = \frac{[\mathbf{M}_{QW}(z45^\circ, 0) \mathbf{M}_P(0^\circ)](x45^\circ + \varepsilon)}{T_{QW}T_P} = \begin{pmatrix} 1 & -xS_{2\varepsilon}D_P & xC_{2\varepsilon}D_P & 0 \\ 0 & c_{2\varepsilon}(c_{2\varepsilon}c_P + zS_{2\varepsilon}S_P)Z_P & s_{2\varepsilon}(c_{2\varepsilon}c_P + zS_{2\varepsilon}S_P)Z_P & x(zS_{2\varepsilon}c_P - c_{2\varepsilon}S_P)Z_P \\ 0 & c_{2\varepsilon}(s_{2\varepsilon}c_P - zC_{2\varepsilon}S_P)Z_P & s_{2\varepsilon}(s_{2\varepsilon}c_P - zC_{2\varepsilon}S_P)Z_P & -x(zC_{2\varepsilon}c_P + s_{2\varepsilon}S_P)Z_P \\ zD_P & -xzs_{2\varepsilon} & xzc_{2\varepsilon} & 0 \end{pmatrix} = \begin{pmatrix} 1 & -xS_{2\varepsilon}D_P & xC_{2\varepsilon}D_P & 0 \\ 0 & c_{2\varepsilon}c_{2\varepsilon-zP}Z_P & s_{2\varepsilon}c_{2\varepsilon-zP}Z_P & xs_{z2\varepsilon-P}Z_P \\ 0 & c_{2\varepsilon}s_{2\varepsilon-zP}Z_P & s_{2\varepsilon}c_{2\varepsilon-zP}Z_P & -xc_{z2\varepsilon-P}Z_P \\ zD_P & -xzs_{2\varepsilon} & xzc_{2\varepsilon} & 0 \end{pmatrix} \quad (\text{S.10.18.3})$$

Circular polariser with QWP with phase shift error ω and ideal linear polariser as $\pm 45^\circ$ calibrator

$$D_P = 1, Z_P = 0, \Rightarrow$$

$$\frac{\mathbf{M}_{CP}(z, \omega, x45^\circ + \varepsilon)}{T_{CP}} = \frac{[\mathbf{M}_{QW}(z45^\circ, \omega) \mathbf{M}_P(0^\circ)](x45^\circ + \varepsilon)}{T_{QW}T_P} = \begin{pmatrix} 1 & -xS_{2\varepsilon} & xC_{2\varepsilon} & 0 \\ xS_{2\varepsilon}S_\omega & -S_{2\varepsilon}^2S_\omega & c_{2\varepsilon}S_{2\varepsilon}S_\omega & 0 \\ -xC_{2\varepsilon}S_\omega & c_{2\varepsilon}S_{2\varepsilon}S_\omega & -c_{2\varepsilon}^2S_\omega & 0 \\ zC_\omega & -xS_{2\varepsilon}zC_\omega & xC_{2\varepsilon}zC_\omega & 0 \end{pmatrix} = \begin{vmatrix} 1 & 1 \\ xS_{2\varepsilon}S_\omega & -xS_{2\varepsilon} \\ -xC_{2\varepsilon}S_\omega & xC_{2\varepsilon} \\ zC_\omega & 0 \end{vmatrix} \quad (\text{S.10.18.4})$$

Ideal circular polariser as $\pm 45^\circ$ calibrator (see Eq. (E.27))

with $D_P = 1$, $\omega = 0 \Rightarrow$

$$\mathbf{M}_{CP}(z) = \mathbf{M}_{QW}(z45^\circ) \mathbf{M}_P(0^\circ) =$$

$$= T_{CP} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z \\ 0 & 0 & 1 & 0 \\ 0 & z & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = T_{CP} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ z & z & 0 & 0 \end{pmatrix} \quad (\text{S.10.18.5})$$

$$= T_{CP} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z \\ 0 & 0 & 1 & 0 \\ 0 & z & 0 & 0 \end{pmatrix} \left| \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array} \right\rangle \left| \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array} \right\rangle = T_{CP} \left| \begin{array}{c} 1 \\ 0 \\ 0 \\ z \end{array} \right\rangle \left| \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array} \right\rangle$$

with $z = \pm 1$

Rotated, ideal circular polariser as $\pm 45^\circ$ calibrator

$D_P = 1$, $\omega = 0 \Rightarrow$

$$\frac{\mathbf{M}_{CP}(z, x45^\circ + \varepsilon)}{T_{CP}} = \frac{\mathbf{R}(x45^\circ + \varepsilon) \mathbf{M}_{QW}(z45^\circ) \mathbf{M}_P(0^\circ) \mathbf{R}(-x45^\circ - \varepsilon)}{T_{CP}} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -xS_{2\varepsilon} & -x\mathbf{c}_{2\varepsilon} & 0 \\ 0 & x\mathbf{c}_{2\varepsilon} & -xS_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z \\ 0 & 0 & 1 & 0 \\ 0 & z & 0 & 0 \end{pmatrix} \left| \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array} \right\rangle \left| \begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array} \right\rangle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -xS_{2\varepsilon} & x\mathbf{c}_{2\varepsilon} & 0 \\ 0 & -x\mathbf{c}_{2\varepsilon} & -xS_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & -xS_{2\varepsilon} & x\mathbf{c}_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ z & -zXS_{2\varepsilon} & zX\mathbf{c}_{2\varepsilon} & 0 \end{pmatrix} \left| \begin{array}{c} 1 \\ 0 \\ 0 \\ z \end{array} \right\rangle \left| \begin{array}{c} 1 \\ -xS_{2\varepsilon} \\ x\mathbf{c}_{2\varepsilon} \\ 0 \end{array} \right\rangle \quad (\text{S.10.18.6})$$

Rotated circular polariser as $+45^\circ$ calibrator with a QWP, with retardation error

From Eq. (S.10.9.3)

$$\frac{\mathbf{M}_{QW}(\phi) \mathbf{M}_P}{T_{QW} T_P} =$$

$$= \begin{pmatrix} 1 + \mathbf{c}_{2\phi} D_{QW} D_P & \mathbf{c}_{2\phi} D_{QW} + D_P & \dots \\ \mathbf{c}_{2\phi} D_{QW} + (1 - S_{2\phi}^2 W_{QW}) D_P & \mathbf{c}_{2\phi} D_{QW} D_P + (1 - S_{2\phi}^2 W_{QW}) & \dots \\ S_{2\phi} (D_{QW} + \mathbf{c}_{2\phi} W_{QW} D_P) & S_{2\phi} (D_{QW} D_P + \mathbf{c}_{2\phi} W_{QW}) & \dots \\ S_{2\phi} Z_{QW} S_{QW} D_P & S_{2\phi} Z_{QW} S_{QW} & \dots \\ \dots & S_{2\phi} D_{QW} Z_P \mathbf{c}_P & S_{2\phi} D_{QW} Z_P S_P \\ \dots & S_{2\phi} (\mathbf{c}_{2\phi} W_{QW} \mathbf{c}_P + Z_{QW} S_{QW} S_P) Z_P & S_{2\phi} (\mathbf{c}_{2\phi} W_{QW} S_P - Z_{QW} S_{QW} \mathbf{c}_P) Z_P \\ \dots & [(1 - \mathbf{c}_{2\phi}^2 W_{QW}) \mathbf{c}_P - \mathbf{c}_{2\phi} Z_{QW} S_{QW} S_P] Z_P & [(1 - \mathbf{c}_{2\phi}^2 W_{QW}) S_P + \mathbf{c}_{2\phi} Z_{QW} S_{QW} \mathbf{c}_P] Z_P \\ \dots & -(\mathbf{c}_{2\phi} S_{QW} \mathbf{c}_P + \mathbf{c}_{QW} S_P) Z_{QW} Z_P & (-\mathbf{c}_{2\phi} S_{QW} S_P + \mathbf{c}_{QW} \mathbf{c}_P) Z_{QW} Z_P \end{pmatrix} \quad (\text{S.10.18.7})$$

$$\mathbf{M}_{QW}(\phi)\mathbf{M}_P = T_{QW}T_P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{c}_{2\phi}^2 & \mathbf{s}_{2\phi}\mathbf{c}_{2\phi} & -\mathbf{s}_{2\phi} \\ 0 & \mathbf{s}_{2\phi}\mathbf{c}_{2\phi} & \mathbf{s}_{2\phi}^2 & \mathbf{c}_{2\phi} \\ 0 & \mathbf{s}_{2\phi} & -\mathbf{c}_{2\phi} & 0 \end{pmatrix} \begin{vmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix} = T_{QW}T_P \begin{vmatrix} 1 & & & \\ & \mathbf{c}_{2\phi}^2 & & \\ & \mathbf{s}_{2\phi}\mathbf{c}_{2\phi} & & \\ & & & 1 \end{vmatrix} \begin{vmatrix} & & & 1 \\ & & & \\ & & & \\ & & & 1 \end{vmatrix} \quad (\text{S.10.18.8})$$

$$\phi = x45^\circ + \varepsilon \Rightarrow \mathbf{c}_{2\phi} \rightarrow -x\mathbf{s}_{2\varepsilon}, \mathbf{s}_{2\phi} \rightarrow x\mathbf{c}_{2\varepsilon} \Rightarrow$$

$$\frac{\mathbf{M}_{QW}(x45^\circ + \varepsilon)\mathbf{M}_P}{T_{QW}T_P} = \begin{pmatrix} 1 - x\mathbf{s}_{2\varepsilon}D_{QW}D_P & -x\mathbf{s}_{2\varepsilon}D_{QW} + D_P & \cdot & \cdot \\ -x\mathbf{s}_{2\varepsilon}D_{QW} + (1 - \mathbf{c}_{2\varepsilon}^2 W_{QW})D_P & -x\mathbf{s}_{2\varepsilon}D_{QW}D_P + (1 - \mathbf{c}_{2\varepsilon}^2 W_{QW}) & \cdot & \cdot \\ \mathbf{c}_{2\varepsilon}(xD_{QW} - \mathbf{s}_{2\varepsilon}W_{QW}D_P) & \mathbf{c}_{2\varepsilon}(xD_{QW}D_P - \mathbf{s}_{2\varepsilon}W_{QW}) & \cdot & \cdot \\ x\mathbf{c}_{2\varepsilon}Z_{QW}S_{QW}D_P & x\mathbf{c}_{2\varepsilon}Z_{QW}S_{QW} & \cdot & \cdot \\ \cdot & \cdot & x\mathbf{c}_{2\varepsilon}D_{QW}Z_P\mathbf{c}_P & x\mathbf{c}_{2\varepsilon}D_{QW}Z_P\mathbf{s}_P \\ \cdot & \cdot & -\mathbf{c}_{2\varepsilon}(\mathbf{s}_{2\varepsilon}W_{QW}\mathbf{c}_P - xZ_{QW}S_{QW}\mathbf{s}_P)Z_P & -\mathbf{c}_{2\varepsilon}(\mathbf{s}_{2\varepsilon}W_{QW}\mathbf{s}_P + xZ_{QW}S_{QW}\mathbf{c}_P)Z_P \\ \cdot & \cdot & [(1 - \mathbf{s}_{2\varepsilon}^2 W_{QW})\mathbf{c}_P + x\mathbf{s}_{2\varepsilon}Z_{QW}S_{QW}\mathbf{s}_P]Z_P & [(1 - \mathbf{s}_{2\varepsilon}^2 W_{QW})\mathbf{s}_P - x\mathbf{s}_{2\varepsilon}Z_{QW}S_{QW}\mathbf{c}_P]Z_P \\ \cdot & \cdot & -(-x\mathbf{s}_{2\varepsilon}\mathbf{s}_{QW}\mathbf{c}_P + \mathbf{c}_{QW}\mathbf{s}_P)Z_{QW}Z_P & (+x\mathbf{s}_{2\varepsilon}\mathbf{s}_{QW}\mathbf{s}_P + \mathbf{c}_{QW}\mathbf{c}_P)Z_{QW}Z_P \end{pmatrix} \quad (\text{S.10.18.9})$$

$$\phi = 45^\circ + \varepsilon \Rightarrow \mathbf{c}_{2\phi} \rightarrow -\mathbf{s}_{2\varepsilon}, \mathbf{s}_{2\phi} \rightarrow \mathbf{c}_{2\varepsilon} \Rightarrow$$

$$\frac{\mathbf{M}_{QW}(45^\circ + \varepsilon)\mathbf{M}_P}{T_{QW}T_P} = \begin{pmatrix} 1 - \mathbf{s}_{2\varepsilon}D_{QW}D_P & -\mathbf{s}_{2\varepsilon}D_{QW} + D_P & \cdot & \cdot \\ -\mathbf{s}_{2\varepsilon}D_{QW} + (1 - \mathbf{c}_{2\varepsilon}^2 W_{QW})D_P & -\mathbf{s}_{2\varepsilon}D_{QW}D_P + (1 - \mathbf{c}_{2\varepsilon}^2 W_{QW}) & \cdot & \cdot \\ \mathbf{c}_{2\varepsilon}(D_{QW} - \mathbf{s}_{2\varepsilon}W_{QW}D_P) & \mathbf{c}_{2\varepsilon}(D_{QW}D_P - \mathbf{s}_{2\varepsilon}W_{QW}) & \cdot & \cdot \\ \mathbf{c}_{2\varepsilon}Z_{QW}S_{QW}D_P & \mathbf{c}_{2\varepsilon}Z_{QW}S_{QW} & \cdot & \cdot \\ \cdot & \cdot & \mathbf{c}_{2\varepsilon}D_{QW}Z_P\mathbf{c}_P & \mathbf{c}_{2\varepsilon}D_{QW}Z_P\mathbf{s}_P \\ \cdot & \cdot & -\mathbf{c}_{2\varepsilon}(\mathbf{s}_{2\varepsilon}W_{QW}\mathbf{c}_P - Z_{QW}S_{QW}\mathbf{s}_P)Z_P & -\mathbf{c}_{2\varepsilon}(\mathbf{s}_{2\varepsilon}W_{QW}\mathbf{s}_P + Z_{QW}S_{QW}\mathbf{c}_P)Z_P \\ \cdot & \cdot & [(1 - \mathbf{s}_{2\varepsilon}^2 W_{QW})\mathbf{c}_P + \mathbf{s}_{2\varepsilon}Z_{QW}S_{QW}\mathbf{s}_P]Z_P & [(1 - \mathbf{s}_{2\varepsilon}^2 W_{QW})\mathbf{s}_P - \mathbf{s}_{2\varepsilon}Z_{QW}S_{QW}\mathbf{c}_P]Z_P \\ \cdot & \cdot & (\mathbf{s}_{2\varepsilon}\mathbf{s}_{QW}\mathbf{c}_P - \mathbf{c}_{QW}\mathbf{s}_P)Z_{QW}Z_P & (\mathbf{s}_{2\varepsilon}\mathbf{s}_{QW}\mathbf{s}_P + \mathbf{c}_{QW}\mathbf{c}_P)Z_{QW}Z_P \end{pmatrix} \quad (\text{S.10.18.10})$$

$$D_Q = 0, \mathbf{c}_{QW} = 0, \mathbf{s}_{QW} = 1, Z_{QW} = 1, W_{QW} = (1 - \mathbf{c}_{QW}) = 1 \Rightarrow$$

$$\frac{[\mathbf{M}_{QW}(45^\circ + \varepsilon)\mathbf{M}_P](\mathbf{x}45^\circ)}{T_{QW}T_P} = \begin{pmatrix} 1 & D_P & \dots & \\ s_{2\varepsilon}^2 D_P & s_{2\varepsilon}^2 & \dots & \\ -\mathbf{c}_{2\varepsilon} s_{2\varepsilon} D_P & -\mathbf{c}_{2\varepsilon} s_{2\varepsilon} D_P & \dots & \\ \mathbf{c}_{2\varepsilon} D_P & \mathbf{c}_{2\varepsilon} & \dots & \\ \dots & 0 & \dots & 0 \\ \dots & -\mathbf{c}_{2\varepsilon} (s_{2\varepsilon} \mathbf{c}_P - s_P) Z_P & -\mathbf{c}_{2\varepsilon} (s_{2\varepsilon} s_P + \mathbf{c}_P) Z_P & \\ \dots & (\mathbf{c}_{2\varepsilon}^2 \mathbf{c}_P + s_{2\varepsilon} s_P) Z_P & (\mathbf{c}_{2\varepsilon}^2 s_P - s_{2\varepsilon} \mathbf{c}_P) Z_P & \\ \dots & s_{2\varepsilon} \mathbf{c}_P Z_P & s_{2\varepsilon} s_P Z_P & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\mathbf{x} & 0 \\ 0 & \mathbf{x} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\mathbf{x} D_P & 0 \\ s_{2\varepsilon}^2 D_P & (\mathbf{c}_{2\varepsilon}^2 \mathbf{c}_P + s_{2\varepsilon} s_P) Z_P & \mathbf{c}_{2\varepsilon} s_{2\varepsilon} D_P & \mathbf{x} (\mathbf{c}_{2\varepsilon}^2 s_P - s_{2\varepsilon} \mathbf{c}_P) Z_P \\ -\mathbf{c}_{2\varepsilon} s_{2\varepsilon} D_P & \mathbf{c}_{2\varepsilon} (s_{2\varepsilon} \mathbf{c}_P - s_P) Z_P & s_{2\varepsilon}^2 & -\mathbf{x} \mathbf{c}_{2\varepsilon} (s_{2\varepsilon} s_P + \mathbf{c}_P) Z_P \\ \mathbf{c}_{2\varepsilon} D_P & \mathbf{x} s_{2\varepsilon} \mathbf{c}_P Z_P & -\mathbf{x} \mathbf{c}_{2\varepsilon} & s_{2\varepsilon} s_P Z_P \end{pmatrix} \quad (\text{S.10.18.11})$$

$$\left. \begin{aligned} \phi = 45^\circ + \varepsilon \Rightarrow \mathbf{c}_{2\phi} \rightarrow -s_{2\varepsilon}, s_{2\phi} \rightarrow \mathbf{c}_{2\varepsilon}, \Delta_{QW} = 90^\circ + \zeta \Rightarrow \mathbf{c}_{QW} = -s_\zeta \rightarrow 0, \mathbf{s}_{QW} = \mathbf{c}_\zeta \rightarrow 1 \\ (1 - \mathbf{c}_{2\varepsilon}^2 W_{QW}) = (1 - \mathbf{c}_{2\varepsilon}^2 (1 - Z_{QW} \mathbf{c}_{QW})) = s_{2\varepsilon}^2 + \mathbf{c}_{2\varepsilon}^2 Z_{QW} \mathbf{c}_{QW} = (s_{2\varepsilon}^2 - \mathbf{c}_{2\varepsilon}^2 Z_{QW} s_\zeta), \\ (1 - s_{2\varepsilon}^2 W_{QW}) = \mathbf{c}_{2\varepsilon}^2 + s_{2\varepsilon}^2 Z_{QW} \mathbf{c}_{QW} = (\mathbf{c}_{2\varepsilon}^2 - s_{2\varepsilon}^2 Z_{QW} s_\zeta) \end{aligned} \right\} \Rightarrow$$

$$\frac{[\mathbf{M}_{QW}(45^\circ + \varepsilon, \Delta_{QW} = 90^\circ + \zeta)\mathbf{M}_P]}{T_{QW}T_P} = \begin{pmatrix} 1 - s_{2\varepsilon} D_{QW} D_P & -s_{2\varepsilon} D_{QW} + D_P & \dots & \\ -s_{2\varepsilon} D_{QW} + (s_{2\varepsilon}^2 - \mathbf{c}_{2\varepsilon}^2 Z_{QW} s_\zeta) D_P & -s_{2\varepsilon} D_{QW} D_P + (s_{2\varepsilon}^2 - \mathbf{c}_{2\varepsilon}^2 Z_{QW} s_\zeta) & \dots & \\ \mathbf{c}_{2\varepsilon} (D_{QW} - s_{2\varepsilon} (1 - Z_{QW} \mathbf{c}_{QW})) D_P & \mathbf{c}_{2\varepsilon} (D_{QW} D_P - s_{2\varepsilon} (1 - Z_{QW} \mathbf{c}_{QW})) & \dots & \\ \mathbf{c}_{2\varepsilon} Z_{QW} \mathbf{c}_\zeta D_P & \mathbf{c}_{2\varepsilon} Z_{QW} \mathbf{c}_\zeta & \dots & \\ \dots & \mathbf{c}_{2\varepsilon} D_{QW} Z_P \mathbf{c}_P & \dots & \mathbf{c}_{2\varepsilon} D_{QW} Z_P s_P \\ \dots & -\mathbf{c}_{2\varepsilon} (s_{2\varepsilon} (1 - Z_{QW} \mathbf{c}_{QW}) \mathbf{c}_P - Z_{QW} \mathbf{c}_\zeta s_P) Z_P & -\mathbf{c}_{2\varepsilon} (s_{2\varepsilon} (1 - Z_{QW} \mathbf{c}_{QW}) s_P + Z_{QW} \mathbf{c}_\zeta \mathbf{c}_P) Z_P & \\ \dots & [(\mathbf{c}_{2\varepsilon}^2 - s_{2\varepsilon}^2 Z_{QW} s_\zeta) \mathbf{c}_P + s_{2\varepsilon} Z_{QW} \mathbf{c}_\zeta s_P] Z_P & [(\mathbf{c}_{2\varepsilon}^2 - s_{2\varepsilon}^2 Z_{QW} s_\zeta) s_P - s_{2\varepsilon} Z_{QW} \mathbf{c}_\zeta \mathbf{c}_P] Z_P & \\ \dots & (s_{2\varepsilon} \mathbf{c}_\zeta \mathbf{c}_P + s_\zeta s_P) Z_{QW} Z_P & (s_{2\varepsilon} \mathbf{c}_\zeta s_P - s_\zeta \mathbf{c}_P) Z_{QW} Z_P & \end{pmatrix} \quad (\text{S.10.18.12})$$

$$\phi = 45^\circ \Rightarrow \mathbf{c}_{2\phi} = 0, \mathbf{s}_{2\phi} = 1 \Rightarrow$$

$$\frac{\mathbf{M}_{QW}(45^\circ)\mathbf{M}_P}{T_{QW}T_P} = \begin{pmatrix} 1 & D_P & \mathbf{c}_P D_{QW} Z_P & \mathbf{s}_P D_{QW} Z_P \\ \mathbf{c}_{QW} Z_{QW} D_P & \mathbf{c}_{QW} Z_{QW} & \mathbf{s}_{QW} \mathbf{s}_P Z_{QW} Z_P & -\mathbf{s}_{QW} \mathbf{c}_P Z_{QW} Z_P \\ D_{QW} & D_{QW} D_P & \mathbf{c}_P Z_P & \mathbf{s}_P Z_P \\ \mathbf{s}_{QW} Z_{QW} D_P & Z_{QW} \mathbf{s}_{QW} & -\mathbf{c}_{QW} \mathbf{s}_P Z_{QW} Z_P & \mathbf{c}_{QW} \mathbf{c}_P Z_{QW} Z_P \end{pmatrix} \quad (\text{S.10.18.13})$$

$$D_{QW} = 0, Z_{QW} = 1, W_{QW} = (1 - \mathbf{c}_{QW}), \phi = 45^\circ + \varepsilon \Rightarrow \mathbf{c}_{2\phi} \rightarrow -\mathbf{s}_{2\varepsilon}, \mathbf{s}_{2\phi} \rightarrow \mathbf{c}_{2\varepsilon} \Rightarrow$$

$$\frac{\mathbf{M}_{CP}(0^\circ)}{T_{CP}} = \frac{\mathbf{M}_{QW}(45^\circ)\mathbf{M}_P}{T_{QW}T_P} = \begin{pmatrix} 1 & D_P & 0 & 0 \\ \mathbf{c}_{QW} D_P & \mathbf{c}_{QW} & \mathbf{s}_{QW} \mathbf{s}_P Z_P & -\mathbf{s}_{QW} \mathbf{c}_P Z_P \\ 0 & 0 & \mathbf{c}_P Z_P & \mathbf{s}_P Z_P \\ \mathbf{s}_{QW} D_P & \mathbf{s}_{QW} & -\mathbf{c}_{QW} \mathbf{s}_P Z_P & \mathbf{c}_{QW} \mathbf{c}_P Z_P \end{pmatrix} \quad (\text{S.10.18.14})$$

$$\phi = 45^\circ + \varepsilon \Rightarrow \mathbf{c}_{2\phi} \rightarrow -\mathbf{s}_{2\varepsilon}, \mathbf{s}_{2\phi} \rightarrow \mathbf{c}_{2\varepsilon}, \Delta_{QW} = 90^\circ + \zeta \Rightarrow \mathbf{c}_{QW} = -\mathbf{s}_\zeta, \mathbf{s}_{QW} = \mathbf{c}_\zeta \Rightarrow$$

$$\left[\frac{\mathbf{M}_{QW}(45^\circ, \Delta_{QW} = 90^\circ + \zeta)\mathbf{M}_P}{T_{QW}T_P} \right] = \begin{pmatrix} 1 & D_P & \mathbf{c}_P D_{QW} Z_P & \mathbf{s}_P D_{QW} Z_P \\ -\mathbf{s}_\zeta Z_{QW} D_P & -\mathbf{s}_\zeta Z_{QW} & \mathbf{c}_\zeta \mathbf{s}_P Z_{QW} Z_P & -\mathbf{c}_\zeta \mathbf{c}_P Z_{QW} Z_P \\ D_{QW} & D_{QW} D_P & \mathbf{c}_P Z_P & \mathbf{s}_P Z_P \\ \mathbf{c}_\zeta Z_{QW} D_P & \mathbf{c}_\zeta Z_{QW} & \mathbf{s}_\zeta \mathbf{s}_P Z_{QW} Z_P & -\mathbf{s}_\zeta \mathbf{c}_P Z_{QW} Z_P \end{pmatrix} \quad (\text{S.10.18.15})$$

$$\left[\frac{\mathbf{M}_{QW}(45^\circ, \Delta_Q = 90^\circ + \zeta)\mathbf{M}_P}{T_{QW}T_P} \right] (\mathbf{x}45^\circ + \varepsilon) =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathbf{x}\mathbf{s}_{2\varepsilon} & -\mathbf{x}\mathbf{c}_{2\varepsilon} & 0 \\ 0 & \mathbf{x}\mathbf{c}_{2\varepsilon} & -\mathbf{x}\mathbf{s}_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & D_P & \mathbf{c}_P D_{QW} Z_P & \mathbf{s}_P D_{QW} Z_P \\ -\mathbf{s}_\zeta Z_{QW} D_P & -\mathbf{s}_\zeta Z_{QW} & \mathbf{c}_\zeta \mathbf{s}_P Z_{QW} Z_P & -\mathbf{c}_\zeta \mathbf{c}_P Z_{QW} Z_P \\ D_{QW} & D_{QW} D_P & \mathbf{c}_P Z_P & \mathbf{s}_P Z_P \\ \mathbf{c}_\zeta Z_{QW} D_P & \mathbf{c}_\zeta Z_{QW} & \mathbf{s}_\zeta \mathbf{s}_P Z_{QW} Z_P & -\mathbf{s}_\zeta \mathbf{c}_P Z_{QW} Z_P \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathbf{x}\mathbf{s}_{2\varepsilon} & \mathbf{x}\mathbf{c}_{2\varepsilon} & 0 \\ 0 & -\mathbf{x}\mathbf{c}_{2\varepsilon} & -\mathbf{x}\mathbf{s}_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & -\mathbf{x}(\mathbf{s}_{2\varepsilon} D_P + \mathbf{c}_{2\varepsilon} \mathbf{c}_P D_{QW} Z_P) & \dots & \dots \\ \mathbf{x}(\mathbf{s}_{2\varepsilon} \mathbf{s}_\zeta Z_{QW} D_P - \mathbf{c}_{2\varepsilon} D_{QW}) & -\mathbf{s}_{2\varepsilon}^2 \mathbf{s}_\zeta Z_{QW} + \mathbf{c}_{2\varepsilon}^2 \mathbf{c}_P Z_P + \mathbf{c}_{2\varepsilon} \mathbf{s}_{2\varepsilon} (\mathbf{c}_\zeta \mathbf{s}_P Z_P Z_{QW} + D_{QW} D_P) & \dots & \dots \\ -\mathbf{x}(\mathbf{c}_{2\varepsilon} \mathbf{s}_\zeta Z_{QW} D_P + \mathbf{s}_{2\varepsilon} D_{QW}) & \mathbf{c}_{2\varepsilon} \mathbf{s}_{2\varepsilon} (\mathbf{s}_\zeta Z_{QW} + \mathbf{c}_P Z_P) - \mathbf{c}_{2\varepsilon}^2 \mathbf{c}_\zeta \mathbf{s}_P Z_P Z_{QW} + \mathbf{s}_{2\varepsilon}^2 D_{QW} D_P & \dots & \dots \\ \mathbf{c}_\zeta Z_{QW} D_P & -\mathbf{x}(\mathbf{s}_{2\varepsilon} \mathbf{c}_\zeta + \mathbf{c}_{2\varepsilon} \mathbf{s}_\zeta \mathbf{s}_P Z_P) Z_{QW} & \dots & \dots \\ \dots & \mathbf{x}(\mathbf{c}_{2\varepsilon} D_P - \mathbf{s}_{2\varepsilon} \mathbf{c}_P D_{QW} Z_P) & \dots & \dots \\ \dots & \mathbf{c}_{2\varepsilon} \mathbf{s}_{2\varepsilon} (\mathbf{s}_\zeta Z_{QW} + \mathbf{c}_P Z_P) + \mathbf{s}_{2\varepsilon}^2 \mathbf{c}_\zeta \mathbf{s}_P Z_P Z_{QW} - \mathbf{c}_{2\varepsilon}^2 D_{QW} D_P & \dots & \dots \\ \dots & -\mathbf{x}(\mathbf{c}_{2\varepsilon} \mathbf{s}_\zeta Z_{QW} + \mathbf{s}_{2\varepsilon}^2 \mathbf{c}_P Z_P - \mathbf{c}_{2\varepsilon} \mathbf{s}_{2\varepsilon} (\mathbf{c}_\zeta \mathbf{s}_P Z_P Z_{QW} + D_{QW} D_P)) & \dots & \dots \\ \dots & \mathbf{x}(\mathbf{c}_{2\varepsilon} \mathbf{c}_\zeta - \mathbf{s}_{2\varepsilon} \mathbf{s}_\zeta \mathbf{s}_P Z_P) Z_{QW} & \dots & \dots \end{pmatrix} \quad (\text{S.10.18.16})$$

CP with ideal LP: $D_p = 1, Z_p = 0, \Rightarrow$

$$\begin{aligned}
& \frac{[\mathbf{M}_{QW}(45^\circ, \Delta_{QW}=90^\circ + \zeta) \mathbf{M}_P](x45^\circ + \varepsilon)}{T_{QW}T_P} = \\
& = \begin{pmatrix} 1 & -x s_{2\varepsilon} & x c_{2\varepsilon} & 0 \\ x(s_{2\varepsilon} s_\zeta Z_{QW} - c_{2\varepsilon} D_{QW}) & -s_{2\varepsilon}^2 s_\zeta Z_{QW} + c_{2\varepsilon} s_{2\varepsilon} D_{QW} & c_{2\varepsilon} s_{2\varepsilon} s_\zeta Z_{QW} - c_{2\varepsilon}^2 D_{QW} & 0 \\ -x(c_{2\varepsilon} s_\zeta Z_{QW} + s_{2\varepsilon} D_{QW}) & c_{2\varepsilon} s_{2\varepsilon} s_\zeta Z_{QW} + s_{2\varepsilon}^2 D_{QW} & -c_{2\varepsilon}^2 s_\zeta Z_{QW} - c_{2\varepsilon} s_{2\varepsilon} D_{QW} & 0 \\ c_\zeta Z_{QW} & -x s_{2\varepsilon} c_\zeta Z_{QW} & x c_{2\varepsilon} c_\zeta Z_{QW} & 0 \end{pmatrix} = \\
& = \begin{pmatrix} 1 & & & \\ x(s_{2\varepsilon} s_\zeta Z_{QW} - c_{2\varepsilon} D_{QW}) & & & \\ -x(c_{2\varepsilon} s_\zeta Z_{QW} + s_{2\varepsilon} D_{QW}) & & & \\ c_\zeta Z_{QW} & & & \end{pmatrix} \begin{pmatrix} 1 \\ -x s_{2\varepsilon} \\ x c_{2\varepsilon} \\ 0 \end{pmatrix} \\
\end{aligned} \tag{S.10.18.17}$$

CP before the polarising beam-splitter:

$$\begin{aligned}
& \frac{\langle \mathbf{A}_S(y) | \mathbf{M}_{CP}(x45^\circ + \varepsilon, \omega) \rangle}{T_S T_{CP}} = \frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{M}_{CP}(x45^\circ + \varepsilon, \omega) \rangle}{T_S T_{CP}} = \\
& = \left\langle \begin{pmatrix} 1 \\ y D_S \\ 0 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 & & & \\ x(s_{2\varepsilon} s_\omega Z_{QW} - c_{2\varepsilon} D_{QW}) & & & \\ -x(c_{2\varepsilon} s_\omega Z_{QW} + s_{2\varepsilon} D_{QW}) & & & \\ c_\omega Z_{QW} & & & \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ -x s_{2\varepsilon} \\ x c_{2\varepsilon} \\ 0 \end{pmatrix} = [1 + xy D_S (s_{2\varepsilon} s_\omega Z_{QW} - c_{2\varepsilon} D_{QW})] \left\langle \begin{pmatrix} 1 \\ -x s_{2\varepsilon} \\ x c_{2\varepsilon} \\ 0 \end{pmatrix} \right\rangle \\
\end{aligned} \tag{S.10.18.18}$$

$$\begin{aligned}
& \frac{I_S}{\eta_S T_S T_{CP} T_O F_{11} T_E I_L} = \frac{\langle \mathbf{A}_S(y) | \mathbf{M}_{CP}(x45^\circ + \varepsilon, \omega) | \mathbf{I}_{in} \rangle}{T_S T_{CP}} = \\
& = [1 + xy D_S (s_{2\varepsilon} s_\omega Z_{QW} - c_{2\varepsilon} D_{QW})] \left\langle \begin{pmatrix} 1 \\ -x s_{2\varepsilon} \\ x c_{2\varepsilon} \\ 0 \end{pmatrix} \middle| \begin{pmatrix} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{pmatrix} \right\rangle = \\
\end{aligned} \tag{S.10.18.19}$$

$$= [1 + xy D_S (s_{2\varepsilon} s_\omega Z_{QW} - c_{2\varepsilon} D_{QW})] [i_{in} - x(s_{2\varepsilon} q_{in} - c_{2\varepsilon} u_{in})]$$

$D_T = +1, D_R = -1 \Rightarrow$

$$\frac{\eta^*}{\eta} = \frac{1}{\eta} \frac{I_R(x45^\circ + \varepsilon)}{I_T(x45^\circ + \varepsilon)} = \frac{1 - xy (s_{2\varepsilon} s_\omega Z_{QW} - c_{2\varepsilon} D_{QW})}{1 + xy (s_{2\varepsilon} s_\omega Z_{QW} - c_{2\varepsilon} D_{QW})} \tag{S.10.18.20}$$

$$D_Q = 0, Z_Q = 1 \Rightarrow$$

$$\begin{aligned} \frac{\langle \mathbf{A}_S(y) | \mathbf{M}_{CP}(x45^\circ + \varepsilon, \omega) \rangle}{T_S T_{CP}} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y | \mathbf{M}_{CP}(x45^\circ + \varepsilon, \omega) \rangle}{T_S T_{CP}} = \\ &= \left\langle \begin{array}{c} 1 \\ yD_S \\ 0 \\ 0 \end{array} \left\| \begin{array}{c} 1 \\ xS_{2\varepsilon}S_\omega \\ -x\mathbf{c}_{2\varepsilon}S_\omega \\ \mathbf{c}_\omega \end{array} \right. \right. \left. \left. \begin{array}{c} 1 \\ -xS_{2\varepsilon} \\ x\mathbf{c}_{2\varepsilon} \\ 0 \end{array} \right. \right\rangle = [1 + xyD_S S_{2\varepsilon} S_\omega] \left\langle \begin{array}{c} 1 \\ -xS_{2\varepsilon} \\ x\mathbf{c}_{2\varepsilon} \\ 0 \end{array} \right. \left. \right\rangle \end{aligned} \quad (\text{S.10.18.21})$$

CP before the receiving optics:

$$\begin{aligned} \frac{\langle \mathbf{A}_S(y, \gamma) | \mathbf{M}_{CP}(x45^\circ + \varepsilon) \rangle}{T_S T_O T_{CP}} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(\gamma) | \mathbf{M}_{CP}(x45^\circ + \varepsilon) \rangle}{T_S T_O T_{CP}} = \\ &= \left\langle \begin{array}{c} 1 + y\mathbf{c}_{2\gamma} D_S D_O \\ \mathbf{c}_{2\gamma} D_O + yD_S (1 - s_{2\gamma}^2 W_O) \\ s_{2\gamma} (D_O + y\mathbf{c}_{2\gamma} D_S W_O) \\ -y s_{2\gamma} D_S Z_O S_O \end{array} \left\| \begin{array}{c} 1 \\ x(s_{2\varepsilon} s_\varepsilon Z_{QW} - \mathbf{c}_{2\varepsilon} D_{QW}) \\ -x(\mathbf{c}_{2\varepsilon} s_\varepsilon Z_{QW} + s_{2\varepsilon} D_{QW}) \\ \mathbf{c}_\varepsilon Z_{QW} \end{array} \right. \right. \left. \left. \begin{array}{c} 1 \\ -xS_{2\varepsilon} \\ x\mathbf{c}_{2\varepsilon} \\ 0 \end{array} \right. \right\rangle = \end{aligned} \quad (\text{S.10.18.22})$$

$$\begin{aligned} &= 1 + yD_S (\mathbf{c}_{2\gamma} D_O - s_{2\gamma} Z_O S_O \mathbf{c}_\varepsilon Z_{QW}) + \\ &+ x \left\{ \begin{array}{l} [\mathbf{c}_{2\gamma} D_O + yD_S (1 - s_{2\gamma}^2 W_O)] (s_{2\varepsilon} s_\varepsilon Z_{QW} - \mathbf{c}_{2\varepsilon} D_{QW}) - \\ -s_{2\gamma} (D_O + y\mathbf{c}_{2\gamma} D_S W_O) (\mathbf{c}_{2\varepsilon} s_\varepsilon Z_{QW} + s_{2\varepsilon} D_{QW}) \end{array} \right\} \end{aligned}$$

$$\gamma = 0 \Rightarrow$$

$$\begin{aligned} \frac{\langle \mathbf{A}_S(y, 0) | \mathbf{M}_{CP}(x45^\circ + \varepsilon) \rangle}{T_S T_O T_{CP}} &= \frac{\langle \mathbf{M}_S \mathbf{R}_y \mathbf{M}_O(0) | \mathbf{M}_{CP}(x45^\circ + \varepsilon) \rangle}{T_S T_O T_{CP}} = \\ &= 1 + yD_S D_O + x \left\{ [D_O + yD_S] (s_{2\varepsilon} s_\varepsilon Z_{QW} - \mathbf{c}_{2\varepsilon} D_{QW}) \right\} \end{aligned} \quad (\text{S.10.18.23})$$

CP with QWP without diattenuation: $D_{QW} = 0$, $Z_{QW} = 1$, \Rightarrow

$$\frac{[\mathbf{M}_{QW}(45^\circ, \Delta_{QW}=90^\circ + \zeta) \mathbf{M}_P](x45^\circ + \varepsilon)}{T_{QW}T_P} = \begin{pmatrix} 1 & -xs_{2\varepsilon}D_P & \cdot & \cdot \\ xs_{2\varepsilon}s_\zeta D_P & -s_{2\varepsilon}^2s_\zeta + c_{2\varepsilon}(c_{2\varepsilon}c_P + s_{2\varepsilon}c_\zeta s_P)Z_P & \cdot & \cdot \\ -xc_{2\varepsilon}s_\zeta D_P & c_{2\varepsilon}s_{2\varepsilon}s_\zeta + c_{2\varepsilon}(s_{2\varepsilon}c_P - c_{2\varepsilon}c_\zeta s_P)Z_P & \cdot & \cdot \\ c_\zeta D_P & -x(s_{2\varepsilon}c_\zeta + c_{2\varepsilon}s_\zeta s_P Z_P) & \cdot & \cdot \\ \cdot & \cdot & xc_{2\varepsilon}D_P & 0 \\ \cdot & \cdot & c_{2\varepsilon}s_{2\varepsilon}s_\zeta + s_{2\varepsilon}(c_{2\varepsilon}c_P + s_{2\varepsilon}c_\zeta s_P)Z_P & x(s_{2\varepsilon}c_\zeta c_P - c_{2\varepsilon}s_P)Z_P \\ \cdot & \cdot & -c_{2\varepsilon}^2s_\zeta + s_{2\varepsilon}(s_{2\varepsilon}c_P - c_{2\varepsilon}c_\zeta s_P)Z_P & -x(c_{2\varepsilon}c_\zeta c_P + s_{2\varepsilon}s_P)Z_P \\ \cdot & \cdot & x(c_{2\varepsilon}c_\zeta - s_{2\varepsilon}s_\zeta s_P Z_P) & -s_\zeta c_P Z_P \end{pmatrix} \quad (\text{S.10.18.24})$$

CP with QW without diattenuation and phase shift error: $D_{QW} = 0$, $Z_{QW} = 1$, $\zeta = 0 \Rightarrow$

$$\frac{[\mathbf{M}_{QW}(45^\circ, \Delta_{QW}=90^\circ) \mathbf{M}_P](x45^\circ + \varepsilon)}{T_{QW}T_P} = \begin{pmatrix} 1 & -xs_{2\varepsilon}D_P & xc_{2\varepsilon}D_P & 0 \\ 0 & c_{2\varepsilon}c_{2\varepsilon-P}Z_P & s_{2\varepsilon}c_{2\varepsilon-P}Z_P & xs_{2\varepsilon-P}Z_P \\ 0 & c_{2\varepsilon}s_{2\varepsilon-P}Z_P & s_{2\varepsilon}s_{2\varepsilon-P}Z_P & -xc_{2\varepsilon-P}Z_P \\ D_P & -xs_{2\varepsilon} & xc_{2\varepsilon} & 0 \end{pmatrix} \quad (\text{S.10.18.25})$$

S.10.19 Circular analyser (CA)

(see Chipman (2009a) Chap. 15.26)

In order to keep the same flexibility regarding the mutual orientation between the linear polariser and the $\lambda/4$ plate as for the circular polariser, we construct the ideal circular analyser with a $\lambda/4$ plate at $\pm 45^\circ$ and a linear polariser at 0° or 90° (according to Chipman (2009a), Sect. 15.18: left circular analyser for $x, z = +1$) from Eqs. (S.10.8.5) and (S.10.17.4)

$$\frac{\mathbf{M}_{CA}(x, z)}{T_{CA}} = \frac{\mathbf{M}_P(45^\circ - x45^\circ)}{T_P} \frac{\mathbf{M}_{QW}(z45^\circ)}{T_{QW}} = \begin{pmatrix} 1 & \backslash & / & 1 \\ x & \backslash & / & x \\ 0 & \backslash & / & 0 \\ 0 & \backslash & / & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z \\ 0 & 0 & 1 & 0 \\ 0 & z & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \backslash & / & 1 \\ x & \backslash & / & 0 \\ 0 & \backslash & / & 0 \\ 0 & \backslash & / & -xz \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -zx \\ x & 0 & 0 & -z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{S.10.19.1})$$

with $x, z = \pm 1$

with $zx = +1 \Rightarrow$ left circ. analyzer

with $zx = -1 \Rightarrow$ right circ. analyzer

Ideal circular analyser with QWP at $z45^\circ$ with error angle ε

$$\begin{aligned}
\frac{\mathbf{M}_{CA}(x,z,\varepsilon)}{T_{CA}} &= \mathbf{R}(+\varepsilon) \frac{\mathbf{M}_{CA}(x,z)}{T_{CA}} \mathbf{R}(-\varepsilon) = \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\varepsilon} & -xs_{2\varepsilon} & 0 \\ 0 & xs_{2\varepsilon} & c_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -zx \\ x & 0 & 0 & -z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\varepsilon} & xs_{2\varepsilon} & 0 \\ 0 & -xs_{2\varepsilon} & c_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \\
&= \begin{pmatrix} 1 & 0 & 0 & -zx \\ xc_{2\varepsilon} & 0 & 0 & -zc_{2\varepsilon} \\ s_{2\varepsilon} & 0 & 0 & -zxs_{2\varepsilon} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{array}{c|c|c} 1 & & 1 \\ xc_{2\varepsilon} & & 0 \\ s_{2\varepsilon} & & 0 \\ 0 & & -zx \end{array}
\end{aligned} \tag{S.10.19.2}$$

S.11 Helpful relations

(see also S.10.10)

$$\frac{1-\delta}{1+\delta} = a = \frac{I_{\parallel} - I_{\perp}}{I_{\parallel} + I_{\perp}}, \quad \delta = \frac{1-a}{1+a} \tag{S.11.1}$$

$$D_T \equiv \frac{T_T^p - T_T^s}{T_T^p + T_T^s}, \quad T_T \equiv 0.5(T_T^p + T_T^s) \tag{S.11.2}$$

$$\left. \begin{aligned} 1 - D_T &= 1 - \frac{T_T^p - T_T^s}{T_T^p + T_T^s} = \frac{2T_T^s}{T_T^p + T_T^s} = \frac{T_T^s}{T_T} \\ 1 + D_T &= 1 - \frac{T_T^p - T_T^s}{T_T^p + T_T^s} = \frac{2T_T^p}{T_T^p + T_T^s} = \frac{T_T^p}{T_T} \end{aligned} \right\} \Rightarrow \frac{1 + D_T}{1 - D_T} = \frac{T_T^p}{T_T^s} \tag{S.11.3}$$

$$\begin{aligned} 1 + D_O D_T &= 1 + \frac{T_O^p - T_O^s}{T_O^p + T_O^s} \frac{T_T^p - T_T^s}{T_T^p + T_T^s} = \frac{(T_O^p + T_O^s)(T_T^p + T_T^s) + (T_O^p - T_O^s)(T_T^p - T_T^s)}{(T_O^p + T_O^s)(T_T^p + T_T^s)} = \\ &= \frac{T_O^p T_T^p + T_O^s T_T^s}{2T_O T_T} \end{aligned} \tag{S.11.4}$$

$$\begin{aligned} 1 - D_O D_T &= 1 - \frac{T_O^p - T_O^s}{T_O^p + T_O^s} \frac{T_T^p - T_T^s}{T_T^p + T_T^s} = \frac{(T_O^p + T_O^s)(T_T^p + T_T^s) - (T_O^p - T_O^s)(T_T^p - T_T^s)}{(T_O^p + T_O^s)(T_T^p + T_T^s)} = \\ &= \frac{T_O^s T_T^p + T_O^p T_T^s}{2T_O T_T} \end{aligned}$$

$$D_O = 0 \Rightarrow Z_O = \sqrt{1 - D_O^2} = 1, \quad W_O = 1 - c_O \tag{S.11.5}$$

$$|D_O| = 1 \Rightarrow Z_O = \sqrt{1 - D_O^2} = 0, \quad W_O = 1 \tag{S.11.6}$$

S.12 Trigonometric relations

$$\left. \begin{aligned} s_\phi c_\phi &= \frac{1}{2} s_{2\phi} \\ c_\alpha c_\beta &= \frac{1}{2} (c_{\alpha-\beta} + c_{\alpha+\beta}) \\ s_\alpha s_\beta &= \frac{1}{2} (c_{\alpha-\beta} - c_{\alpha+\beta}) \end{aligned} \right\} \Rightarrow \begin{cases} c_\phi c_\beta + s_\phi s_\beta = \frac{1}{2} (c_{\phi-\beta} + c_{\phi+\beta} + c_{\phi-\beta} - c_{\phi+\beta}) = c_{\phi-\beta} \\ c_\phi c_\beta - s_\phi s_\beta = \frac{1}{2} (c_{\phi-\beta} + c_{\phi+\beta} - c_{\phi-\beta} + c_{\phi+\beta}) = c_{\phi+\beta} \\ s_\phi c_\beta - c_\phi s_\beta = \frac{1}{2} (s_{\phi-\beta} + s_{\phi+\beta} - s_{\phi+\beta} + s_{\phi-\beta}) = s_{\phi-\beta} \\ s_\phi c_\beta + c_\phi s_\beta = \frac{1}{2} (s_{\phi-\beta} + s_{\phi+\beta} + s_{\phi+\beta} - s_{\phi-\beta}) = s_{\phi+\beta} \end{cases} \quad (\text{S.12.1})$$

with $\phi = x45^\circ + \varepsilon, x = \pm 1 \Rightarrow$

$$c_{2\phi} = \cos[2(x45^\circ + \varepsilon)] = \cos(\pm 90^\circ + 2\varepsilon) = \mp \sin(2\varepsilon) = -x s_{2\varepsilon}$$

$$s_{2\phi} = \sin[2(x45^\circ + \varepsilon)] = \sin(\pm 90^\circ + 2\varepsilon) = \pm \cos(2\varepsilon) = x c_{2\varepsilon}$$

with $\phi = x45^\circ + 45 + \varepsilon, x = \pm 1 \Rightarrow$

$$c_{2\phi} = \cos[2(x45^\circ + 45 + \varepsilon)] = \cos(\pm 90^\circ + 90^\circ + 2\varepsilon) = \mp \cos(2\varepsilon) = -x c_{2\varepsilon} \quad (\text{S.12.2})$$

$$s_{2\phi} = \sin[2(x45^\circ + 45 + \varepsilon)] = \sin(\pm 90^\circ + 90^\circ + 2\varepsilon) = \mp \sin(2\varepsilon) = -x s_{2\varepsilon}$$

$$\text{for } (x45^\circ + \varepsilon) \rightarrow (x45^\circ + 45 + \varepsilon) \Rightarrow \begin{cases} -x s_{2\varepsilon} \rightarrow -x c_{2\varepsilon} \\ x c_{2\varepsilon} \rightarrow -x s_{2\varepsilon} \end{cases}$$

with $x = \pm 1 \Rightarrow$

$$\cos[2(x45^\circ)] = 0$$

$$\sin[2(x45^\circ)] = x$$

$$\cos[2(x45^\circ + 45)] = -x$$

$$\sin[2(x45^\circ + 45)] = 0 \quad (\text{S.12.3})$$

$$\cos[2(-x45^\circ - \gamma)] = \cos(\mp 90^\circ - 2\gamma) = \mp \sin(2\gamma) = -x s_{2\gamma}$$

$$\sin[2(-x45^\circ - \gamma)] = \sin(\mp 90^\circ - 2\gamma) = \mp \cos(2\gamma) = -x c_{2\gamma}$$

$$\cos\{2[x(45^\circ + \gamma)]\} = \cos[\pm(90^\circ + 2\gamma)] = -\sin(2\gamma) = -s_{2\gamma}$$

$$\sin\{2[x(45^\circ + \gamma)]\} = \sin[\pm(90^\circ + 2\gamma)] = \pm \cos(2\gamma) = x c_{2\gamma}$$

$\phi = x22.5^\circ + \varepsilon/2, x = \pm 1 \Rightarrow$

$$c_{4\phi} = \cos[4(x22.5^\circ + \varepsilon/2)] = \cos(x90^\circ + 2\varepsilon) = -x s_{2\varepsilon} \quad (\text{S.12.4})$$

$$s_{4\phi} = \sin[4(x22.5^\circ + \varepsilon/2)] = \sin(x90^\circ + 2\varepsilon) = x c_{2\varepsilon}$$

$$s_{4\phi} = 2s_{2\phi}c_{2\phi}$$

$$c_{4\phi} = c_{2\phi}^2 - s_{2\phi}^2 = 2c_{2\phi}^2 - 1 = 1 - 2s_{2\phi}^2 \Rightarrow \begin{cases} 1 + c_{4\phi} = 2c_{2\phi}^2 \\ 1 - c_{4\phi} = 2s_{2\phi}^2 \end{cases} \quad (\text{S.12.5})$$

$$\begin{aligned}
1 - c_{2\varepsilon}^2 W_P &= 1 - (1 - s_{2\varepsilon}^2) W_P = 1 - W_P + s_{2\varepsilon}^2 W_P = Z_P c_P + s_{2\varepsilon}^2 W_P = Z_P c_P + s_{2\varepsilon}^2 (1 - Z_P c_P) \Rightarrow \\
1 - c_{2\varepsilon}^2 W_P &= s_{2\varepsilon}^2 + c_{2\varepsilon}^2 Z_P c_P \\
1 - s_{2\varepsilon}^2 W_P &= c_{2\varepsilon}^2 + s_{2\varepsilon}^2 Z_P c_P
\end{aligned} \tag{S.12.6}$$

S.12.1 Tangent half-angle substitution

The substitution Eq. (S.12.1.1) is sometimes called Weierstrass substitution, but it can already be found in Euler's Institutionum calculi integralis (Eneström number E342: Vol. 1 Part 1, Sect. 1, Chap. 5, Problem 29, <http://eulerarchive.maa.org/pages/E342.html>).

$$t = \tan\left(\frac{\theta}{2}\right) \Leftrightarrow \sin \theta = \frac{2t}{1+t^2} \tag{S.12.1.1}$$

With this substitution we can write Eq. (S.12.1.2) and yield ε from Eq. (S.12.1.3). For small ε we get the approximation Eq. (S.12.1.4).

$$t = Ks_{2\varepsilon} = \tan\left(\frac{\theta}{2}\right) \Leftrightarrow \sin \theta = Y = \frac{2Ks_{2\varepsilon}}{1+(Ks_{2\varepsilon})^2} \tag{S.12.1.2}$$

$$\varepsilon = \frac{1}{2} \arcsin \left[\frac{1}{K} \tan \left(\frac{\arcsin(Y)}{2} \right) \right] \tag{S.12.1.3}$$

$$\begin{aligned}
K < 1 \wedge \varepsilon \ll 1 &\Rightarrow \\
\varepsilon &\approx \frac{Y}{2K}
\end{aligned} \tag{S.12.1.4}$$

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